Score:

Name: Solutions

**Period (circle one): 1 2 3 4 5 6**
**Team (circle one): a b c d e f**

# SM286 – Final Examination– Spring 2011

**Approximate all Answers to 2 Decimal Places**

1. Radioactive decay is governed by the 1st order differential equation $\frac{dM}{dt} = kM$. A scrap of paper taken from the Dead Sea scrolls contains .795 times the original mass Carbon-14 ($^{14}C$) that it had when the scrolls were written? If $^{14}C$ has a half life of ~~5720~~ 5700 years how old are the scrolls (note: use DE methods to solve this problem).

$$\int \frac{dM}{M} = \int k\,dt \Rightarrow \ln|M| = kt + c \Rightarrow M = e^{kt+c} \Rightarrow \boxed{M = Ae^{kt}}$$

$$M(0) = A = M_0 \Rightarrow \boxed{M = M_0 e^{kt}}$$

$$M(5720) = \cancel{M_0} e^{k(5720)} = \frac{1}{2}\cancel{M_0}$$

$$\Rightarrow k = \frac{1}{5700}\ln\left(\frac{1}{2}\right) \approx -.000122$$

$$\Rightarrow M = M_0 e^{-.000122t}$$

$$\Rightarrow .795\cancel{M_0} = \cancel{M_0} e^{-.000122t}$$

$$t = \ln(.795)/-.000122 \approx \boxed{1887. \text{ yrs}}$$

2. The equation for current in a simple CR series circuit is $L\frac{d^2q}{dt^2} + \frac{1}{C}q = V$, where $q$ is charge, $L$ is inductance, $C$ is capacitance, V is the voltage, and $t$ is time.

   a. Solve for charge if the circuit consists of a 5 Henry inductor, ~~.05~~ .1 farad capacitor, and a 10 volt battery. Assume that the initial charge $q(0) = 0$ coulombs and the initial current $i(0) = 4$ amps (recall that $i = \frac{dq}{dt}$).

   b. Combine the "sin/cos" part of your answer in part a as a single expression $Asin(\omega t + \phi)$.

$$5q'' + \frac{1}{.1}q = 10 \Rightarrow 5q'' + 10q = 10$$

$$\Rightarrow \boxed{q'' + 2q = 2}$$

① $q_H \Rightarrow \quad (D^2+2)q_H = 0 \Rightarrow \quad D = \pm\sqrt{2}i \Rightarrow q_H = C_1\sin(\sqrt{2}t) + C_2\cos(\sqrt{2}t)$

② $q_P = A \Rightarrow q_P' = q_P'' = 0 \Rightarrow 0 + 2A = 2 \Rightarrow A = 1$

③ no conflicts!!

④ $q = C_1\sin(\sqrt{2}t) + C_2\cos(\sqrt{2}t) + 1$

⑤ $q(0) = C_2 + 1 = 0 \Rightarrow C_2 = -1$

$q = C_1\sin(\sqrt{2}t) - 1\cos(\sqrt{2}t) + 1$

$q' = \sqrt{2}C_1\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t)$ ⓐ

$q'(0) = \sqrt{2}C_1 = 4 \Rightarrow C_1 = \frac{4}{\sqrt{2}} = 2\sqrt{2}$ $\boxed{q = 2\sqrt{2}\sin(\sqrt{2}t) - 1\cos(\sqrt{2}t) + 1}$

$A = \sqrt{(2\sqrt{2})^2 + 1^2} = \sqrt{8+1} = \sqrt{9} = 3$

$\phi = \tan^{-1}(-1/2\sqrt{2}) = \tan^{-1}(-\sqrt{2}/4) = -.33$ Radians or $-19.47^\circ$

ⓑ $\Rightarrow \boxed{q = 3\sin(\sqrt{2}t - .33) + 1}$

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3. Solve the differential equation $x\frac{dy}{dx} - 4y = x^6e^x$ where $y(1) = \cancel{0}\,6$

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x \Rightarrow \quad u(x) = e^{\int -\frac{4}{x}dx} = e^{-4\ln x}$$

$$= e^{\ln(x^{-4})} = x^{-4}$$

$$\int \frac{d}{dx}\left[x^{-4}y\right] = \int xe^x\,dx$$

$$\Rightarrow \quad x^{-4}y = xe^x - e^x + C$$

$$\Rightarrow \quad y = x^5e^x - x^4e^x + Cx^4$$

$$y(1) = e - e + C = 6 \Rightarrow C = 6$$

$$\boxed{y = x^5e^x - x^4e^x + 6x^4}$$

CHECK:

$$y' = 5x^4e^x + x^5e^x - 4x^3e^x - x^4e^x + 24x^3$$

$$xy' = \cancel{5x^5e^x} + x^6e^x - \cancel{4x^4e^x} - \cancel{x^5e^x} + \cancel{24x^4}$$

$$-\,4y = \cancel{4x^5e^x} - \cancel{4x^4e^x} + \cancel{24x^4}$$

$$= x^6e^x \quad \checkmark\checkmark$$

4. Solve the following system of D.E.'s $\begin{cases} \frac{dx}{dt} = -2x + 2y \\ \frac{dy}{dt} = \quad x - 3y \end{cases}$ with initial conditions $\begin{cases} x(0) = 4 \\ y(0) = 5 \end{cases}$.

Using your solution, calculate $x(.2)$ and $y(.2)$.

$$X' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad X' = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} X \Rightarrow \left| \begin{bmatrix} -2-\lambda & 2 \\ 1 & -3-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow 6 + 5\lambda + \lambda^2 - 2 = 0 \Rightarrow \lambda^2 + 5\lambda + 4 = 0 \Rightarrow (\lambda+4)(\lambda+1) = 0$$

$$\lambda = -4, \lambda = -1 \Rightarrow$$

$$\lambda_1 = -4 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \quad x_1 = -x_2 \Rightarrow V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \quad \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = 2x_2 \Rightarrow V_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow X = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \Rightarrow \begin{matrix} x = -c_1 e^{-4t} + 2c_2 e^{-t} \\ y = c_1 e^{-4t} + c_2 e^{-t} \end{matrix}$$

$$\Rightarrow \quad \begin{aligned} x(0) &= -c_1 + 2c_2 = 4 \\ y(0) &= c_1 + c_2 = 5 \\ & 3c_2 = 9 \Rightarrow c_2 = 3, c_1 = 2 \end{aligned}$$

$$\boxed{\begin{aligned} x &= -2e^{-4t} + 6e^{-t} \\ y &= 2e^{-4t} + 3e^{-t} \end{aligned}}$$

$$\boxed{\begin{aligned} x(.2) &= 4.01 \\ y(.2) &= 3.35 \end{aligned}}$$

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5. Using the system of equations in Problem 4:
   a. Use Euler's method to estimate the value of $x(.2)$ using a step size of $\Delta t = .1$.
   b. Why does this answer differ from the answer in Problem 4?
   c. How might you improve this answer?

| $t$ | $x$ | $y$ | $\frac{dx}{dt}$ | $\frac{dy}{dt}$ | $\Delta t$ | $\Delta x$ | $\Delta y$ |
|---|---|---|---|---|---|---|---|
| 0 | 4 | 5 | 2 | -11 | .1 | .2 | 1.1 |
| .1 | 4.2 | 3.9 | -.6 | -7.5 | -.06 | -.06 | -.75 |
| .2 | 4.14 | 3.15 | | | | | |

x(.2) y(.2)

(a)

(b) Different because this is a numeric approximation

(c) Make $\Delta t$ smaller.

6. Using matrices, balance the chemical formula:

$$\underline{A}\,HClO_4 + \underline{B}\,P_4O_{10} \rightarrow \underline{C}\,H_3PO_4 + \underline{D}\,Cl_2O_7$$

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & B \end{array} \\ \begin{array}{c} H \\ Cl \\ O \\ P \end{array} & \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 4 & 10 & -4 & -7 & 0 \\ 0 & 4 & -1 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{RREF}$$

$$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1/6 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$A - 2D = 0 \Rightarrow A = 2D$

$B - \frac{1}{6}D = 0 \Rightarrow B = \frac{1}{6}D$

$C - \frac{2}{3}D = 0 \Rightarrow C = \frac{2}{3}D$

let $D = 6$,

$A = 12$, $B = 1$, $C = 4$

$\therefore$ $\boxed{12\,HClO_4 + P_4O_{10} \rightarrow 4\,H_3PO_4 + 6\,Cl_2O_7}$

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7. Using the definition of the Maclaurin series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ :
   a. **Derive** the Maclaurin series for $\frac{1}{1+x}$.
   b. **Using you answer in part a**, derive the Maclaurin series for $\frac{1}{1+\cancel{x^2}}$? $x^3$
   c. **Using you answer in part a**, derive Maclaurin series for $\frac{1}{1-\cancel{x^3}}$? $x^2$

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $f^{(n)}(0)/n!$ |
|---|---|---|---|
| 0 | $(1+x)^{-1}$ | 1 | $1/0! = 1$ |
| 1 | $-(1+x)^{-2}$ | $-1$ | $1/1! = -1$ |
| 2 | $+2(1+x)^{-3}$ | 2 | $2/2! = 1$ |
| 3 | $-6(1+x)^{-4}$ | $-6$ | $-6/3! = -1$ |

a pattern emerges

(a)

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \dots \Rightarrow \boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n}$$

(b) $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^3)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n}}$$

(c) $$\frac{1}{1-x^3} = \frac{1}{1+(-x^3)} = \sum_{n=0}^{\infty} (-1)^n (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^{2n} x^{2n}$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} x^{2n}}$$

8. Solve the differential equation $\frac{d^2x}{dt^2} + t\frac{dx}{dt} = 0$, $x(0) = \cancel{1}\,2$, $x'(0) = \cancel{2}\,1$. (Hint: Let $x = \sum_{n=0}^{\infty} c_n t^n$.) Write out the 1st four terms of the series.

$$\frac{dx}{dt} = \sum_{n=1}^{\infty} c_n(n)t^{n-1} \qquad \frac{d^2x}{dt^2} = \sum_{n=2}^{\infty} c_n(n)(n-1)t^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n(n)(n-1)t^{n-2} + t\sum_{n=1}^{\infty} c_n(n)t^{n-1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} c_n(n)(n-1)t^{n-2} + \sum_{n=1}^{\infty} c_n(n)t^{n} = 0$$

$k = n-2 \Rightarrow n = k+2$ $\qquad n = k$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)t^k + \sum_{k=1}^{\infty} c_k k t^k = 0$$

$$\Rightarrow c_2(2)(1)t^0 + \sum_{n=1}^{\infty}\left[c_{k+2}(k+2)(k+1)t^k + c_k k t^k\right] = 0$$

$\boxed{c_2 = 0}$ $\qquad c_{k+2}(k+2)(k+1) + c_k k = 0 \Rightarrow \boxed{c_{k+2} = \frac{-c_k(k)}{(k+2)(k+1)}}$

since $\begin{cases} x(0) = 1 \Rightarrow c_0 = 2 \\ x'(0) = 2 \Rightarrow c_1 = 1 \end{cases}$

$k=1 \Rightarrow c_3 = \frac{-c_1(1)}{(3)(2)} = (-1)\frac{1}{6} = -1/6$

$k=2 \Rightarrow c_4 = \frac{-c_2(2)}{(4)(3)} = 0\left(\frac{1}{6}\right) = 0$

$k=3 \Rightarrow c_5 = -c_3\left(\frac{3}{(5)(4)}\right) = -\left(-\frac{1}{6}\right)\left(\frac{3}{20}\right) = 1/40$

$$\Rightarrow \boxed{x \approx 2 + t - 1/6\, t^3 + 1/40\, t^5 \ \ldots}$$

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9. The temperature in a one dimension rod is governed by the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ with boundary and initial conditions } \begin{cases} u(0,t) = x(\pi,t) = 0 \\ u(x,0) = (x-\pi)^{2} x^{\cancel{0}2} \end{cases}$$

Assume that the eigenvalues for $\boldsymbol{\lambda} \leq 0$ produce trivial solutions.

a. Find a general solution $u(x,t)$.

~~b. Write out the 1st three non-zero terms of $u(x,t)$.~~

$$u = XT \Rightarrow \frac{XT'}{XT} = \frac{X''T}{XT} \Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$\Rightarrow T' + \lambda T = 0 \Rightarrow T = ae^{-\lambda t}$$

$$X'' + \lambda X = 0 \Rightarrow X = c_1 \sin(\lambda^{1/2} x) + c_2 \cos(\lambda^{1/2} x)$$

$$\Rightarrow X(0) = c_2 = 0 \Rightarrow X(\pi) = \sin(\lambda^{1/2}\pi) = 0$$

$$\Rightarrow \lambda^{1/2}\cancel{\pi} = n\cancel{\pi} \Rightarrow \lambda = n^2$$

$$\Rightarrow X_n = c_n \sin(nx) \qquad T_n = a_n e^{-n^2 t}$$

$$\Rightarrow u_n = X_n T_n = \underset{\uparrow}{b_n} e^{-n^2 t} \sin(nx)$$

$$\Rightarrow u = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx) \Rightarrow u(x,0) = \sum_{n=0}^{\infty} b_n \sin(nx) = (x-\pi) x^3$$

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} (x-\pi)^2 x^2 \sin(nx)\,dx = \frac{4(-1)^n\pi}{n^3} - \frac{48(-1)^n}{n^5\pi} - \frac{4\pi}{n^3} + \frac{48}{n^5\pi}$$

$$\Rightarrow \boxed{u = \sum_{n=1}^{\infty} \left[\frac{4\pi}{n^3}((-1)^n - 1) + \frac{48}{n^5\pi}(1-(-1)^n)\right] e^{-n^2 t} \sin(nx)}$$

10. The equation for a vibrating beam is governed by the PDE $\frac{\partial^2 u}{\partial t^2} = -\alpha^2 \frac{\partial^4 u}{\partial x^4}$ where $u$ is the displacement of the beam and $\alpha$ is a constant that embodies the beam properties. Assume that $u(x,t)$ is separable and use separation of variables techniques to express the PDE as two ordinary differential equations. (DO NOT SOLVE THE RESULTING ODES!)

$$u = XT$$

$$\frac{XT''}{-\alpha^2 XT} = \frac{-\cancel{\alpha^2}\, X''''T}{\cancel{-\alpha^2}\, XT}$$

$$\frac{T''}{-\alpha^2 T} = \frac{X''''}{X} = -\lambda$$

$$\Rightarrow \boxed{\begin{aligned} &T'' + \alpha^2\lambda T = 0 \\ &X'''' + \lambda X = 0 \end{aligned}}$$