Homework Solutions

Section 3.1
(89) #13, 17, 21, 25

13) $T_0 = 70^\circ F$
$T_m = 10^\circ F$
$T(.5) = 50^\circ F$

$\frac{dT}{dt} = k(T-10)$ ($10 = T_m$)

$\Rightarrow \frac{dT}{dt} - kT = -10k \Rightarrow u = e^{\int -k\,dt} = e^{-kt}$

$\Rightarrow \frac{d}{dt}\left[e^{-kt}T\right] = -10ke^{-kt}$

$\Rightarrow \int d\left[e^{-kt}T\right] = \int -10ke^{-kt}\,dt$

$\Rightarrow e^{-kt}T = 10e^{-kt} + C \Rightarrow T = 10 + Ce^{kt}$

$\Rightarrow T(0) = 10 + C = 70 \Rightarrow C = 60$

$\Rightarrow T = 10 + 60e^{kt}$

$\Rightarrow T(.5) = 10 + 60e^{.5k} = 50 \Rightarrow e^{.5k} = \frac{40}{60} = \frac{2}{3}$

$\Rightarrow \frac{1}{2}k = \ln\left(\frac{2}{3}\right) \Rightarrow k = 2\ln\left(\frac{2}{3}\right) = \ln\left(\frac{2}{3}\right)^2$

$\therefore T = 10 + 60e^{t\ln\left(\frac{2}{3}\right)^2} = 10 + 60e^{\ln\left(\frac{2}{3}\right)^{2t}}$

$\Rightarrow T = 10 + 60\left(\frac{2}{3}\right)^{2t} \Rightarrow \boxed{T = 10 + 60\left(\frac{4}{9}\right)^t}$

$\Rightarrow T(1) = 10 + 60\left(\frac{4}{9}\right)^1 = 10 + \frac{240}{9} \approx \boxed{36\frac{2}{3}^\circ = T(1)}$

$\Rightarrow 15 = 10 + 60\left(\frac{4}{9}\right)^t \Rightarrow \frac{5}{60} = \frac{1}{12} = \left(\frac{4}{9}\right)^t$

$\Rightarrow \ln\left(\frac{1}{12}\right) = t\ln\left(\frac{4}{9}\right) \Rightarrow \boxed{t \approx 3.06 \text{ min}}$

17) $\frac{dT}{dt} = k(T - T_m)$ ← temp of thermometer; ↑ temp of oven

$T(0) = 70°F$

$T(.5) = 110°F$

$T(1) = 140°F$

$\Rightarrow \frac{dT}{dt} - kT = -kT_m \Rightarrow \mu(t) = e^{-kt}$

$\Rightarrow \int d[e^{-kt}T] = \int -kT_m e^{-kt}\,dt$

$\Rightarrow e^{-kt}T = T_m e^{-kt} + C \Rightarrow \boxed{T = T_m + Ce^{-kt}}$

$T(0) = T_m + C = 70 \Rightarrow T_m = 70 - C$

$T(.5) = T_m + Ce^{-.5k} = 110° \Rightarrow 70 + C(e^{-.5k} - 1) = 110°$

$T(1) = T_m + Ce^{-k} = 145° \Rightarrow 70 + C(e^{-k} - 1) = 145°$

** $\Rightarrow \begin{cases} C(e^{-.5k} - 1) = 40° \Rightarrow C = 40/(e^{-.5k} - 1) \\ C(e^{-k} - 1) = 75° \end{cases}$

$\therefore \frac{(e^{-k} - 1)(40)}{(e^{-.5} - 1)} = 75 \Rightarrow (e^{-k} - 1) = \frac{75}{40}(e^{-.5k} - 1) \cong 1.875(e^{-.5k} - 1)$

$e^{-k} - 1.875e^{-.5k} + .875 = 0 \Rightarrow x = e^{-.5k}$

$\Rightarrow x^2 - 1.875x + .875 = (x - 1)(x - .875) = 0$

$\Rightarrow$

17) $\therefore$ $x=1 \Rightarrow e^{-.5k}=1 \Rightarrow k=0$

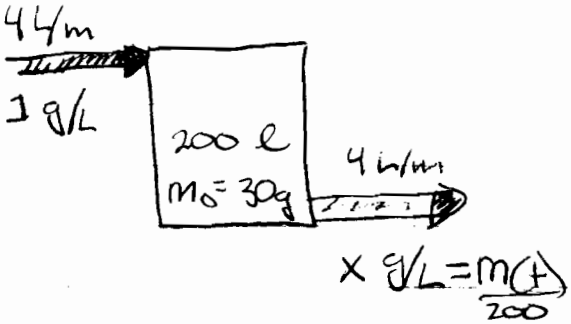
(this is not possible. If $k=0$ there is no heat flux)

$\therefore$ $x=.875 \Rightarrow e^{-.5k}=.875 \Rightarrow k \approx .2671$

$\therefore$ $C(e^{-.2671}-1)=75 \Rightarrow C=-320$

$\Rightarrow T_m = 70 - C \Rightarrow \boxed{T_m = 390^\circ}$

21)



$M(t)$ = grams of salt in tank

$M(0) = 30$

$$\frac{M(t)}{dt} = 4\frac{g}{m} - 4\frac{m(t)}{200}$$

$$\Rightarrow \frac{dM}{dt} + \frac{1}{50}m = 4 \Rightarrow u(t) = e^{\int \frac{1}{50} dt} = e^{.02t}$$

$$\therefore \int \frac{d}{dt}\left[e^{.02t}M\right] = \int 4e^{.02t}\,dt$$

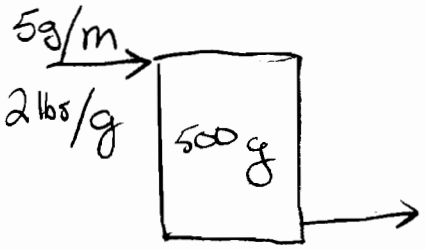
$$\Rightarrow e^{.02t}M = 200e^{.02t} + C$$

$$\Rightarrow M = 200 + Ce^{-.02t}$$

$$\Rightarrow M(0) = 30 \Rightarrow C = -170$$

$$\Rightarrow \boxed{M = 200 - 170e^{-.02t}}$$

25)



$M(t)$ = lbs salt in tank

$M(0) = 0$

$10\,g/m \Rightarrow$ concentration is $\frac{M}{500-5t}$ $\frac{lbs}{gal}$

level is lowering by 5 g/m

$$\frac{dM}{dt} = 10\,lbs/min - \frac{10M}{500-5t}$$

$$\Rightarrow \frac{dM}{dt} + \frac{2}{100-t}M = 10 \quad \Rightarrow \mu(t) = e^{\int \frac{2}{100-t}\,dt}$$

$$= e^{-2\int \frac{1}{t-100}\,dt}$$

$$= e^{-2\ln(t-100)}$$

$$= e^{\ln(t-100)^{-2}}$$

$$= (t-100)^{-2} = \frac{1}{(100-t)^2}$$

$$\int d\left[\frac{1}{(100-t)^2}M\right] = \int \frac{10}{100-t^2}\,dt$$

$$\Rightarrow \frac{1}{(100-t)^2}M = \frac{10}{100-t} + C$$

$$\Rightarrow M = 10(100-t) + C(100-t)^2$$

$$\Rightarrow M(0) = 10(100) + (C)(100)^2 = 0 \Rightarrow C = -1/10$$

$$\therefore \boxed{M = 1000 - 10t - \frac{1}{10}(100-t)^2}$$

Tank is empty in 100 minutes

$\therefore 500 - 5t = 0$ when $t = 100$