HOMEWORK SOLUTION

Section 8.2.1
(324) 1, 3, 5, 7

1) $X = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow X' = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix}$

$\Rightarrow X' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} X$

<u>find</u> <u>eigenvalues</u> $\Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \lambda = 5, \lambda = -1$

<u>find</u> <u>eigenvectors</u>

$\underline{\lambda = 5} \Rightarrow \begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow -4x_1 + 2x_2 = 0$

$\Rightarrow x_2 = 2x_1$

$\Rightarrow \boxed{V_{\lambda=5} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$\underline{\lambda = -1} \Rightarrow \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow 2x_1 + 2x_2 = 0$

$\Rightarrow x_2 = -x_1$

$\boxed{V_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$$\therefore \quad \boxed{X = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{5t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}}$$

I will use this form throughout the solution set.

or $$X = \begin{bmatrix} c_1e^{5t} + c_2e^{-t} \\ 2c_1e^{5t} - c_2e^{-t} \end{bmatrix}$$

or $$\boxed{\begin{aligned} x &= c_1e^{5t} + c_2e^{-t} \\ y &= 2c_1e^{5t} - c_2e^{-t} \end{aligned}}$$

3) $$X' = \begin{bmatrix} -4 & 2 \\ -5/2 & 2 \end{bmatrix} X$$

$$\Rightarrow \det\left(\begin{bmatrix} -4-\lambda & 2 \\ -5/2 & 2-\lambda \end{bmatrix}\right) = 0 \Rightarrow -8 + 4\lambda - 2\lambda + \lambda^2 - (-5) = 0$$

$$\Rightarrow \lambda^2 + 2\lambda - 3 = 0$$

$$\Rightarrow (\lambda+3)(\lambda-1) = 0$$

$$\Rightarrow \lambda = -3,\ \lambda = 1$$

$\Rightarrow$ $\underline{\lambda = -3}$

$$\begin{bmatrix} -1 & 2 \\ -5/2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -x_1 + 2x_2 = 0$$

$$\Rightarrow x_1 = 2x_2$$

$$\Rightarrow V_{\lambda=-3} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 1}$

$$\begin{bmatrix} -5 & 2 \\ -5/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -5x_1 + 2x_2 = 0$$

$$\Rightarrow 2x_2 = 5x_1$$

$$\Rightarrow V_{\lambda=1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\therefore \quad \boxed{X = c_1\begin{pmatrix}2\\1\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}2\\5\end{pmatrix}e^{t}}$$

5) $\det\left(\begin{bmatrix} 10-\lambda & -5 \\ 8 & -12-\lambda \end{bmatrix}\right) = -120-10\lambda+12\lambda+\lambda^2+40=0$

$\Rightarrow \lambda^2+2\lambda-80=0 \Rightarrow (\lambda+10)(\lambda-8)=0 \Rightarrow \lambda=-10, \lambda=8$

$\therefore$ $\underline{\lambda=-10}$

$\begin{bmatrix} 20 & -5 \\ 8 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$

$\Rightarrow 20x_1-5x_2=0$

$\Rightarrow 20x_1=5x_2$

$\Rightarrow x_2=4x_1$

$V_{\lambda=-10}=\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\underline{\lambda=8}$

$\begin{bmatrix} 2 & -5 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$

$\Rightarrow 2x_1-5x_2=0$

$\Rightarrow 2x_1=5x_2$

$\Rightarrow x_2=2 \Rightarrow x_1=5$

$V_{\lambda=8}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\therefore \boxed{X=c_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-10t}+c_2\begin{pmatrix} 5 \\ 2 \end{pmatrix}e^{8t}}$

7) $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad X' = \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix}$

$$X' = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} X = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 \\ 0 & -1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 2-\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)\big((2-\lambda)(-1-\lambda) - 0\big) - 1\underbrace{(\cancel{0-0})}_{0} - 1\underbrace{(\cancel{0-0})}_{0} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(1+\lambda) = 0 \Rightarrow \lambda = 1, \lambda = 2, \lambda = -1$$

: <u>$\lambda = 1$</u>

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow x_2 - x_3 = 0$ $\qquad \Rightarrow \boxed{V_{\lambda=1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

$x_2 = 0$

$x_2 - 2x_3 = 0$

$\Rightarrow x_2 = 0, x_3 = 0$

$x_1 = 1$

actually $x_1$ can be anything sinec the equations don't specify

$\underline{\lambda=2}$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$\Rightarrow -x_1+x_2-x_3=0 \Rightarrow x_1=x_3-x_2 \Rightarrow x_1=2$

$x_2-3x_3=0 \Rightarrow x_2=3x_3 \Rightarrow x_3=1,\ x_2=3$

$$\therefore \boxed{V_{\lambda=2}=\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}$$

$\underline{\lambda=-1}$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$\Rightarrow 2x_1+x_2-x_3=0 \Rightarrow 2x_1=x_3 \Rightarrow x_1=1,\ x_3=2$

$\Rightarrow 3x_2=0 \Rightarrow x_2=0$

$$\therefore \boxed{V_{\lambda=-1}=\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$$

$$\therefore \boxed{X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t}}$$

or

$$\boxed{\begin{aligned} x &= c_1 e^{t} + 2c_2 e^{2t} + c_3 e^{-t} \\ y &= 3c_2 e^{2t} \\ z &= c_2 e^{2t} + 2c_3 e^{-t} \end{aligned}}$$