Homework Solutions

Section 11.1
(402) #1, 10, 12, 17

1) $\int_{-2}^{2} (x)(x^2)\,dx = \int_{-2}^{2} x^3\,dx = \frac{1}{4}x^4\Big|_{-2}^{2} = 4-4 = 0$

There $f(x) = x$ and $g(x) = x^2$ is orthogonal on the interval $[-2, 2]$

10) Use the trig identity

$\Rightarrow \sin\alpha \sin B = \frac{1}{2}\cos(\alpha - B) - \frac{1}{2}\cos(\alpha + B)$

$\Rightarrow \int_0^p \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right) dx$

$= \int_0^p \frac{1}{2}\cos\left(\frac{(n-m)\pi x}{p}\right) - \frac{1}{2}\cos\left(\frac{(n+m)\pi x}{p}\right) dx$

$= \frac{1}{2}\frac{p}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{p}\right) - \frac{1}{2}\frac{p}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{p}\right)\Big|_0^p$

$= \frac{1}{2}\frac{p}{(n-m)\pi}\underbrace{\sin((n-m)\pi)}_{\to 0} - \frac{1}{2}\frac{p}{(n+m)\pi}\underbrace{\sin((n+m)\pi)}_{\to 0}$

$- \frac{1}{2}\frac{p}{(n-m)\pi}\underbrace{\sin(0)}_{\to 0} - \frac{1}{2}\frac{p}{(n+m)\pi}\underbrace{\sin(0)}_{\to 0} = \underline{0}$ ✓

since n, m are integers, $\sin((n-m)\pi) = 0$ and $\sin((n+m)\pi) = 0$

To find norm use trig id:

$$\sin^2(x) = \tfrac{1}{2}(1-\cos(2x))$$

$$\therefore \int_0^p \sin^2\left(\frac{n\pi x}{p}\right)dx = \frac{1}{2}\int_0^p \left(1-\cos\left(\frac{2n\pi x}{p}\right)\right)dx$$

$$= \frac{1}{2}\left[x - \frac{p}{2n\pi}\sin\left(\frac{2n\pi x}{p}\right)\right]\Big|_0^p$$

$$= \frac{1}{2}\left[p - \frac{p}{2n\pi}\underbrace{\cancel{\sin(2n\pi)}}_{\to 0} - (0 - \underbrace{\cancel{\sin(0)}}_{\to 0})\right] = \boxed{\frac{p}{2}}$$

12) Must test 3 Integrals

① $$\int_{-p}^{p}(1)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{p}{n\pi}\sin\left(\frac{n\pi x}{p}\right)\Big|_{-p}^{p}$$

$$= \frac{p}{n\pi}\left(\underbrace{\cancel{\sin(n\pi)}}_{\to 0} - \underbrace{\cancel{\sin(-n\pi)}}_{\to 0}\right] = 0 \checkmark\checkmark$$

② $$\int_{-p}^{p}(1)\left(\sin\left(\frac{n\pi x}{p}\right)\right)dx = -\frac{p}{n\pi}\cos\left(\frac{n\pi x}{p}\right)\Big|_{-p}^{p}$$

$$= -\frac{p}{n\pi}\left[\cos(n\pi) - \cos(-n\pi)\right] = 0 \checkmark\checkmark$$

$\Rightarrow$ (since $\cos(n\pi) = \cos(-n\pi)$)

(3) $\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right) dx$

Use trig id:

$$\sin(\alpha)\cos(B) = \frac{1}{2}\sin(\alpha - B) + \frac{1}{2}\sin(\alpha + B)$$

$$= \int_{-p}^{p} \frac{1}{2}\sin\left(\frac{(m-n)\pi x}{p}\right) + \frac{1}{2}\sin\left(\frac{(m+n)\pi x}{p}\right) dx$$

$$= -\frac{1}{2}\left[\frac{p}{(m-n)\pi}\cos\left(\frac{(m-n)\pi x}{p}\right) + \frac{p}{(m+n)\pi}\cos\left(\frac{(m+n)\pi x}{p}\right)\right]\Bigg|_{-p}^{p}$$

$$= \frac{-1}{2}\left[\frac{p}{(m-n)\pi}\cos((m-n)\pi) + \frac{p}{(m+n)\pi}\cos((m+n)\pi)\right.$$

$$\left. - \frac{p}{(m-n)\pi}\cos(-(m-n)\pi) - \frac{p}{(m+n)\pi}\cos(-(m+n)\pi)\right] = 0 \checkmark$$

(since $\cos(\alpha) = \cos(-\alpha)$)

$\therefore$ set is orthogonal on interval $[-p, p]$

For Norm:

$$\int_{-p}^{p} \sin^2\left(\frac{n\pi x}{p}\right) dx = \frac{1}{2}\int_{-p}^{p} \left(1-\cos\left(\frac{2n\pi x}{p}\right)\right) dx$$

$$= \frac{1}{2}\left[x - \sin\left(\frac{2n\pi x}{p}\right)\right]\Big|_{-p}^{p}$$

$$= \frac{1}{2}\left[p - \cancel{\sin(2n\pi)} - (-p) + \cancel{\sin(2n\pi)}\right]$$

$$= \frac{1}{2}(2p) = p$$

17) $$\|\phi_m(x) + \phi_n(x)\|^2 = \int_a^b (\phi_n(x)+\phi_m(x))(\phi_n(x)+\phi_m(x))\,dx$$

$$= \int_a^b \left[\phi_n^2(x) + 2\phi_n(x)\phi_m(x) + \phi_m^2(x)\right] dx$$

$$= \int_a^b \phi_n^2(x)\,dx + 2\int_a^b \phi_n(x)\phi_m(x)\,dx + \int_a^b \phi_m^2(x)\,dx$$

definition — because $\phi_n \perp \phi_m$ orthogonal — definition

$$= \|\phi_n(x)\|^2 + 0 + \|\phi_m(x)\|^2$$

$$= \|\phi_n(x)\|^2 + \|\phi_m(x)\|^2$$