Homework Solutions

Section 11.2
(407) 1, 3, 5, 9

1) $p = \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \underbrace{\cancel{0}\,dx}_{\to 0} + \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}(\pi) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\cancel{\pi}}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cancel{(0)\cos(nx)}^{\to 0}\,dx + \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\frac{1}{n}\sin(nx)\Big|_0^{\pi} = \frac{1}{\pi}\frac{1}{n}(0-0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{\cancel{\pi}}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\underbrace{\cancel{(0)\sin(nx)}}_{\to 0}\,dx + \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx$$

$$= -\frac{1}{n\pi}\cos(nx)\Big|_0^{\pi} = -\frac{1}{n\pi}\left(\cos(n\pi) - \cos(0)\right)$$

$$= -\frac{1}{n\pi}\left((-1)^n - 1\right) = \frac{1}{n\pi}\left[(-1)^{n+1} + 1\right]$$

↑ i.e. distribute the '−' sign

$$\therefore \boxed{f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[(-1)^{n+1}+1\right]\sin(nx)}$$

3) $p=1$

$$\Rightarrow a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_0^1 x\,dx$$

$$= x\Big|_{-1}^{0} + \tfrac{1}{2}x^2\Big|_0^1 = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$

$$\Rightarrow a_n = \int_{-1}^{0} \cos(n\pi x)\,dx + \int_0^1 x\cos(n\pi x)\,dx$$

$$= \underbrace{\cancel{\frac{\sin(n\pi)}{n\pi}}}_{\to 0} + \frac{\cos(n\pi)}{n^2\pi^2} + \underbrace{\cancel{\frac{\sin(n\pi)}{n\pi}}}_{\to 0} - \frac{1}{n^2\pi^2}$$

$$\cos(n\pi) = (-1)^n$$

$$= \frac{1}{n^2\pi^2}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ -\frac{2}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$\Rightarrow b_n = \int_{-1}^{0} \sin(n\pi x)\,dx + \int_0^1 x\sin(n\pi x)\,dx$$

$$= \cancel{\frac{\cos(n\pi)}{(n\pi)}} - \frac{1}{n\pi} + \underbrace{\cancel{\frac{\sin(n\pi)}{n^2\pi^2}}}_{\to 0} - \cancel{\frac{\cos(n\pi)}{n\pi}} = -\frac{1}{n\pi}$$

$$\therefore \boxed{f(x) = \underbrace{\frac{3}{4}}_{a_0/2} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)\right)}$$

5) $P = \pi$ note: from $-\pi$ to 0, $f(x) = 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\frac{1}{3}x^3\Big|_0^{\pi} = \boxed{\frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx =$$

$$a_n = \frac{1}{\cancel{\pi}}\,\frac{2\cancel{n\pi}\,\cancel{\cos(n\pi)}^{\,(-1)^n} + (n^2\pi^2-2)\,\cancel{\sin(n\pi)}^{\,0}}{n^3}$$

$$= \boxed{\frac{2(-1)^n}{n^3}}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin(nx)\,dx$$

$$= \frac{-\pi\cos(n\pi)}{n} + \frac{2\cos(n\pi)}{n^3\pi} + \frac{2\cancel{\sin(n\pi)}^{\,0}}{n^2} - \frac{2}{n^3\pi}$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}\left((-1)^n - 1\right)$$

$\frac{-\pi}{1} + \frac{2}{\pi}(-2)$

$$\therefore\ \boxed{f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos(nx) + \left(\frac{-\pi}{n}(-1)^n + \frac{2}{n^3\pi}\left((-1)^n-1\right)\right)\sin(nx)\right]}$$

7) $a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \begin{cases} 0 & n=1 \\ \left(\frac{1}{2(n+1)\pi} - \frac{1}{2(n-1)\pi}\right)((-1)^n+1) & n \neq 1 \end{cases}$$

$$\Rightarrow \frac{-1}{(n^2-1)\pi}((-1)^n+1)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx = \begin{cases} 0 & n \neq 1 \\ 1/2 & n=1 \end{cases}$$

$$\therefore \boxed{f(x) = \frac{1}{\pi} + \underbrace{\frac{1}{2}\sin(x)}_{n=1} - \sum_{n=2}^{\infty} \frac{(-1)^n+1}{(n^2-1)\pi}\cos(nx)}$$