Homework Solutions

Section 12.1

(436) 1, 3, 5, 11

1) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \Rightarrow$ Let $u(x,y) = X(x)\,Y(y)$ or XY

$$\Rightarrow \frac{\partial}{\partial x}[XY] = \frac{\partial}{\partial y}[XY]$$

$$\Rightarrow X'Y = XY'$$

$$\Rightarrow \frac{X'}{X} = \frac{Y'}{Y} = -\lambda$$ (seperation of variables)

Solve X

$$\Rightarrow \frac{X'}{X} = -\lambda \Rightarrow (X' + \lambda X) = 0 \Rightarrow (D+\lambda)X = 0 \Rightarrow D = -\lambda$$

$$\Rightarrow \boxed{X = c_1 e^{-\lambda x}}$$

Solve Y

$$\frac{Y'}{Y} = -\lambda \Rightarrow \boxed{Y = c_2 e^{-\lambda y}}$$ (same steps as X)

$$\Rightarrow u = XY \Rightarrow c_1 e^{-\lambda x} c_2 e^{-\lambda y} = c_1 c_2 e^{-\lambda(x+y)}$$

$c_1 c_2 = k$ (another constant)

$$\therefore \boxed{u = k e^{-\lambda(x+y)}}$$

3) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \Rightarrow \quad u = XY$

$\Rightarrow X'Y + XY' = XY \Rightarrow \quad \frac{X'Y}{XY} + \frac{XY'}{XY} = \frac{XY}{XY}$

$\Rightarrow \quad \frac{X'}{X} + \frac{Y'}{Y} = 1 \Rightarrow \quad \frac{X'}{X} = 1 - \frac{Y'}{Y} = -\lambda$

$\Rightarrow \quad \frac{X'}{X} = -\lambda \Rightarrow \boxed{X = c_1 e^{-\lambda x}}$ (same as #1)

$\Rightarrow \quad 1 - \frac{Y'}{Y} = -\lambda \Rightarrow Y - Y' = -\lambda Y$

$\Rightarrow \quad Y' - (Y + \lambda Y) = 0 \Rightarrow Y' - (1+\lambda) Y = 0$

$\Rightarrow \quad [D - (1+\lambda)] Y = 0 \Rightarrow D = 1 + \lambda$

$\Rightarrow \boxed{Y = c_2 e^{(1+\lambda) y}}$

$\therefore \quad u = XY = c_1 e^{-\lambda x} c_2 e^{y + \lambda y}$

$\Rightarrow \quad u = \underbrace{c_1 c_2}_{k} e^{-\lambda x + y + \lambda y}$

$\Rightarrow \boxed{u = k e^{y + \lambda (y - x)}}$

5) $u = XY \Rightarrow xX'Y = yXY'$

$$\Rightarrow \frac{xX'Y}{XY} = \frac{yXY'}{XY} \Rightarrow \frac{xX'}{X} = \frac{yY'}{Y} = -\lambda$$

Find $X(x)$

$$\Rightarrow xX' = -\lambda X \Rightarrow xX' + \lambda X = 0 \Rightarrow X' + \frac{\lambda}{x} X = 0$$

$\Rightarrow$ use <u>integrating factors</u> to solve

$$\mu = e^{\int \frac{\lambda}{x} dx} = e^{\lambda \ln(x)} = e^{\ln(x^\lambda)} = x^\lambda$$

$$\Rightarrow x^\lambda X' + \lambda x^{\lambda-1} X = 0 \Rightarrow \frac{d}{dx}\left[x^\lambda X\right] = 0$$

$$\Rightarrow x^\lambda X = c_1 \Rightarrow \boxed{X = c_1 x^{-\lambda}}$$

Find $Y(y)$

$$\Rightarrow \frac{yY'}{Y} = -\lambda \Rightarrow \quad Y = c_2 y^{-\lambda}$$ (same steps as X)

$$\therefore u = c_1 x^{-\lambda} c_2 y^{-\lambda} = \underbrace{c_1 c_2}_{k} (xy)^{-\lambda}$$

$$\Rightarrow \boxed{u = k(xy)^{-\lambda}}$$

11) $\dfrac{\alpha^2 X''T}{\alpha^2 XT} = \dfrac{XT''}{\alpha^2 XT} \Rightarrow \dfrac{X''}{X} = \dfrac{T''}{\alpha^2 T} = -\lambda$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T'' + \alpha^2 \lambda T = 0 \end{cases} \Rightarrow \begin{matrix} (D^2+\lambda)X = 0 \\ (D^2+\alpha^2\lambda)T = 0 \end{matrix}$$

<u>Case 1</u>: $\lambda < 0 \Rightarrow \quad D = \pm\lambda^{1/2}$ for X, $D = \pm\alpha\lambda^{1/2}$ for T

$$\Rightarrow \begin{cases} X = c_1 e^{-\lambda^{1/2}x} + c_2 e^{\lambda^{1/2}x} \\ T = c_3 e^{-\lambda^{1/2}\alpha t} + c_4 e^{\lambda^{1/2}\alpha t} \end{cases}$$

$$\therefore \boxed{u = XT = \left(c_1 e^{-\lambda^{1/2}x} + c_2 e^{\lambda^{1/2}x}\right)\left(c_3 e^{-\lambda^{1/2}\alpha t} + c_4 e^{\lambda^{1/2}\alpha t}\right)}$$

<u>Case 2</u> $\quad \lambda = 0 \Rightarrow D = 0, 0$ for both X & T

$$\therefore \boxed{u = \underbrace{(c_1 x + c_2)}_{X}\underbrace{(c_3 t + c_4)}_{T}}$$

<u>Case 3</u>: $\lambda > 0 \Rightarrow \quad D = \pm\lambda^{1/2} i$ for X, $D = \pm\lambda^{1/2}\alpha i$ for T

$$\therefore \boxed{u = \left(c_1\cos(\lambda^{1/2}x) + c_2\sin(\lambda^{1/2}x)\right)\left(c_3\cos(\lambda^{1/2}\alpha t) + c_4\cos(\lambda^{1/2}\alpha t)\right)}$$