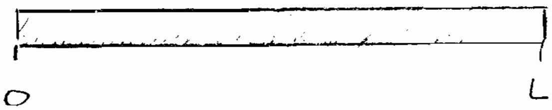
Homework Solutions Section 12.3

(445) #1, 6

(1)



$u(0,t) = u(L,t) = 0$

$$u(x,0) = \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow$ Let $u = XT$

Separate Variables

$\Rightarrow kX''T = XT' \Rightarrow \frac{kX''T}{kXT} = \frac{XT'}{kXT}$

$\Rightarrow \frac{X''}{X} = \frac{T'}{kT} = -\lambda_n$

Solve for $T(t)$

$T' = -k\lambda_n T \Rightarrow T' + k\lambda_n T = 0 \Rightarrow (D + k\lambda_n)T = 0$

$\Rightarrow D = -k\lambda_1$

$\Rightarrow \boxed{T_n = b_n e^{-\lambda_n t}}$

Solve for $X(x)$

$X'' = -\lambda_n X \Rightarrow X'' + \lambda_n X = 0$

$\underline{\lambda = 0} \Rightarrow X'' = 0 \Rightarrow X' = b_1 \Rightarrow X = b_1 x + b_2$

$\Rightarrow X(0) = b_2 = 0 \Rightarrow X = b_1 x \Rightarrow X(L) = b_1 L = 0 \Rightarrow b_1 = 0$

$\Rightarrow X(0) \Rightarrow$ <u>trivial</u>

$\underline{\lambda_n < 0}$ $\quad X'' - \lambda_n X = 0 \Rightarrow (D^2 - \lambda_n)X = 0$

$\Rightarrow D = \pm \lambda_n^{1/2} \quad \therefore X = a_1 e^{-\lambda^{1/2}_n x} + a_2 e^{\lambda^{1/2}_n x}$

$\Rightarrow X(0) = a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$

$X(L) = a_1 e^{-\lambda^{1/2} L} + a_2 e^{\lambda^{1/2} L} = 0$

$\Rightarrow X(L) = a_1 e^{-\lambda^{1/2} L} - a_1 e^{\lambda^{1/2} L} = 0$

$\Rightarrow a_1 (e^{-\lambda^{1/2} L} - e^{\lambda^{1/2} L}) = 0$

this is never zero

$\therefore$ this must be zero

$\Rightarrow a_1 = 0 \Rightarrow a_2 = 0$

$\Rightarrow X = 0, \Rightarrow$ trivial

$\lambda_n > 0 \Rightarrow \quad X'' + \lambda_n X = 0 \Rightarrow (D^2 + \lambda_n) X = 0$

$\therefore D = \pm \lambda_n^{1/2} i \Rightarrow X = a_1 \sin(\lambda_n^{1/2} x) + a_2 \cos(\lambda_n^{1/2} x)$

$X(0) = a_2 = 0 \Rightarrow \quad X = a_1 \sin(\lambda_n^{1/2} x)$

$\Rightarrow X(L) = a_1 \sin(\lambda_n^{1/2} L) = 0$

$\Rightarrow a_1$ can not be zero because this would produce trivial solution

$\therefore \sin(\lambda^{1/2}_n L) = 0 \Rightarrow \lambda^{1/2}_n L = n\pi$

$\Rightarrow \lambda_n^{1/2} = \left(\frac{n\pi}{L}\right)^2$

$$\therefore \quad X_n = a_n \sin\left(\frac{n\pi}{L}x\right)$$

Find u_n

Since $u = XT$

$$u_n = X_n T_n = \underbrace{a_n b_n}_{\text{combine}} e^{-\overset{\lambda_n}{\left(\frac{n\pi}{L}\right)^2} t} \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow u_n = c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

Super-position Principle

$$\Rightarrow u = \sum_{n=1}^{\infty} u_n$$

$$\Rightarrow u = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

Apply Initial Condition

$$u(x, \overset{t=0}{0}) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = \begin{cases} 1 & 0 \le x \le L/2 \\ 0 & L/2 < x < L \end{cases}$$

this is just a sine series

$$\therefore c_n = \frac{2}{L}\int_0^{L/2} (1)\sin\left(\frac{n\pi}{L}x\right)dx + \cancel{\int_{L/2}^{L} 0 \sin\left(\frac{n\pi x}{L}\right)dx}^{\,0}$$

$$\Rightarrow c_n = \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\Rightarrow \boxed{u = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)}$$

6) $\begin{cases} k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

let $u = XT$

Sep Var

$\Rightarrow k X''T - hXT = XT'$

$\Rightarrow \frac{\cancel{k} X'' \cancel{T}}{\cancel{k} X \cancel{T}} - \frac{h \cancel{XT}}{k \cancel{XT}} = \frac{\cancel{X} T'}{k \cancel{X} T} \Rightarrow \frac{X''}{X} - \frac{h}{k} = \frac{T'}{kT} = -\lambda_n$

$\Rightarrow \frac{X''}{X} = \frac{T'}{kT} + \frac{h}{k} = -\lambda_n$

solve for T

$T' + hT = -\lambda_n kT \Rightarrow T' + (h+\lambda_n k)T = 0$

$\Rightarrow (D + (h+\lambda_n k))T = 0 \Rightarrow D = -(h+\lambda_n k)$

$\therefore T_n = b_n e^{-(h+\lambda_n k)t}$

Solve for X

$\frac{X''}{X} = -\lambda_n \Rightarrow X'' + \lambda_n X = 0$

Note! Since boundary conditions are the same as #1 we know that

$X_n = a_n \sin(\frac{n\pi}{L}x)$

Find u

$\therefore u_n = \underbrace{a_n b_n}_{\text{combine } = c_n} e^{-(h+(\frac{n\pi}{L})^2 k)t} \sin(\frac{n\pi}{L}x)$

$$\therefore \boxed{u = \sum_{n=1}^{\infty} c_n e^{-\left(h + \left(\frac{n\pi}{L}\right)^2 k\right)t} \sin\left(\frac{n\pi x}{L}\right)}$$

Apply Initial condition & sine series to find constant

$$\Rightarrow u(x,0) = \sum c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\Rightarrow \boxed{c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$