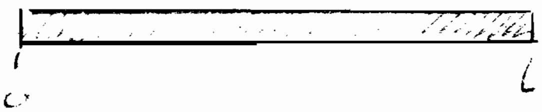
Homework Solutions — Section 12.3 #3, 5

(2)



$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$\left.\frac{du}{dx}\right|_{x=0} = \left.\frac{du}{dx}\right|_{x=L} = 0$$

$$u(x,0) = f(x)$$

Separate Variables

$$\Rightarrow u = XT \Rightarrow kX''T = XT'$$

$$\Rightarrow \frac{\cancel{k}X''\cancel{T}}{\cancel{k}X\cancel{T}} = \frac{\cancel{X}T'}{k\cancel{X}T} \Rightarrow \frac{X''}{X} = \frac{T'}{kT} = -\lambda_n$$

Solve for T

$$\frac{T'}{kT} = -\lambda_n \Rightarrow T' = -\lambda_n T \Rightarrow T' + \lambda_n T = 0$$

$$\Rightarrow \boxed{T_n = b_n e^{-\lambda_n t}}$$

$$X'' = -\lambda_n X \Rightarrow X'' + \lambda_n X = 0$$

$\lambda < 0$

$\underline{\lambda < 0} \rightarrow X'' - \lambda_n X = 0 \Rightarrow (D^2 + \lambda_n)X = 0$

$$\Rightarrow D = \pm\lambda_n^{1/2} \Rightarrow X = q_1 e^{-\lambda_n^{1/2}x} + q_2 e^{+\lambda_n^{1/2}x}$$

$$\Rightarrow X' = -q_1\lambda_n^{1/2} e^{-\lambda_n^{1/2}x} + q_2\lambda_n^{1/2} e^{\lambda_n^{1/2}x}$$

$$\Rightarrow X'(0) = -q_1\lambda_n^{1/2} + q_2\lambda_n^{1/2} = 0 \Rightarrow q_1 = q_2$$

$$\Rightarrow X'(x) = q_1\lambda_n^{1/2}\left(e^{\lambda_n^{1/2}x} - e^{-\lambda_n^{1/2}x}\right)$$

$\lambda<0$

$\Rightarrow X'(L) = a_1 \lambda_n^{1/2}\left[\underbrace{e^{\lambda_n^{1/2}L} - e^{-\lambda_n^{1/2}L}}\right] = 0$

$\lambda_n>0$

This is 0 only if $\lambda_n=0$, but $\lambda_n>0$

therefore $a_1=0 \Rightarrow X=0 \Rightarrow$ trivial

$\underline{\underline{\lambda=0}}$ $\quad X''=0 \Rightarrow X = a_1 x + a_0$

$\Rightarrow X' = a_1 \Rightarrow X'(0) = a_1 = 0 \Rightarrow a_1 = 0$

$\Rightarrow$ this will also satisfy $X(L)=0$

$\therefore X = a_0$ (i.e. a constant)

$\underline{\underline{\lambda>0}}$ $\quad X'' + \lambda_n X = 0 \Rightarrow (D^2+\lambda_n)X = 0$

$\Rightarrow D = \pm \lambda_n^{1/2} i \Rightarrow X = a_1 \sin(\lambda_n^{1/2}x) + a_2\cos(\lambda_n^{1/2}x)$

$\Rightarrow X' = a_1 \lambda_n^{1/2}\cos(\lambda_n^{1/2}x) - a_2\lambda_n^{1/2}\sin(\lambda_n^{1/2}x)$

$\Rightarrow X'(0) = a_1\lambda_n^{1/2} = 0 \Rightarrow a_1 = 0$

$\Rightarrow X = -a_2\lambda_n^{1/2}\sin(\lambda_n^{1/2}x)$

$\Rightarrow X(L) = -a_2\lambda_n^{1/2}\sin(\lambda_n^{1/2}L) = 0$

$\lambda_n<0$

a_2 can't $=0$

$\therefore \sin(\lambda_n^{1/2}L) = 0 \Rightarrow \underline{\lambda_n^{1/2}L = n\pi} \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3\ldots$

$\therefore \boxed{X_n = a_n\cos\left(\frac{n\pi}{L}x\right)}$

$$\therefore \quad u_n = T_n X_x = \underbrace{a_n b_n}_{\uparrow \text{ combine}} e^{-\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$$

Super-position

$$\Rightarrow \boxed{u = \underbrace{c_0}_{\lambda=0 \text{ term}} + \sum_{n=1}^{n} \underbrace{c_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)}_{\lambda_n>0 \text{ terms}}}$$

apply initial condition to find constants

$$u(x,0) = \underbrace{c_0 + \sum_{n=1}^{n} c_n \cos\left(\frac{n\pi}{x}\right)L}_{\uparrow \text{ cosine series}} = f(x)$$

$$\therefore \quad \boxed{\begin{aligned} c_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ c_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

5) $\begin{cases} k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial x}\Big|_{x=0} = \frac{\partial u}{\partial x}\Big|_{x=l} \\ u(x,0) = f(x) \end{cases}$

Sep. Var. $\begin{cases} u = XT \Rightarrow \frac{kX''T}{kXT} - \frac{hXT}{kXT} = \frac{XT'}{kXT} \\ \Rightarrow \frac{X''}{X} - \frac{h}{k} = \frac{T'}{kT} \Rightarrow \frac{X''}{X} = \frac{T'}{kT} + \frac{h}{k} = -\lambda_n \end{cases}$

Solve for T $\begin{cases} \Rightarrow T' + hT = -\lambda_n kT \Rightarrow T' + (h+\lambda_n k)T = 0 \\ \Rightarrow \boxed{T = b_n e^{-(h+\lambda_n k)t}} \end{cases}$

Solve for X $\begin{cases} X'' + \lambda X = 0 \\ \text{we know from (1) that} \\ \underset{\lambda=0}{X = a_0} \quad \& \quad X = \underset{\lambda>0}{a_n} \cos\left(\frac{n\pi}{L}x\right) \end{cases}$

$$\therefore \boxed{u = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(h + \left(\frac{n\pi}{L}\right)^2 k\right)t} \cos\left(\frac{n\pi}{L}x\right)}$$

Apply initial condition as in #1

$$\boxed{\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$