Homework Solutions

Section 12.4
Wave Equation
(448) 1, 2

1) $a^2 \frac{d^2u}{dx^2} = \frac{d^2u}{dt^2} \Rightarrow$ Let $u(x,t) = X(t)T(t)$

$a^2 X''T = XT'' \Rightarrow$ divide by a^2XT

$$\frac{a^2X''T}{a^2XT} = \frac{XT''}{a^2XT} \Rightarrow \frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

$$\Rightarrow \begin{cases} X'' = -\lambda X \\ T'' = -\lambda a^2 T \end{cases} \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T'' + \lambda a^2 T = 0 \end{cases}$$

$\boxed{\text{Skipped Eigenvalue Analysis}}$

$$\Rightarrow X = c_1 \sin(\lambda^{1/2} x) + c_2 \cos(\lambda^{1/2} x)$$

$$T = b_1 \sin(a\lambda^{1/2} t) + b_2 \cos(a\lambda^{1/2} t)$$

Apply Boundary Conditions

$u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow c_1(0) + c_2(1) = 0 \Rightarrow \boxed{c_2 = 0}$

$\therefore X = c_1 \sin(\lambda^{1/2} x)$

$u(L,t) = c_1 \sin(\lambda^{1/2} L) = 0 \Rightarrow \lambda^{1/2} L = n\pi \Rightarrow \lambda^{1/2} = \frac{n\pi}{L}$

$$\therefore X_n = C_n \sin\left(\frac{n\pi}{L} x\right)$$

$$T = b_1 \sin\left(\frac{an\pi}{L} t\right) + b_2 \cos\left(\frac{an\pi}{L} t\right)$$

Apply the '0' Initial Conditions

$$\frac{du}{dt}\bigg|_{t=0} = 0 \Rightarrow T'(0) = 0$$

$$\therefore T' = \frac{an\pi}{L} b_1 \cos\left(\frac{an\pi}{L} t\right) - \frac{an\pi}{L} b_2 \sin\left(\frac{an\pi}{L} t\right)$$

$$T' = \frac{an\pi}{L} b_1 (1) - \frac{an\pi}{L} b_2 (0) = 0 \Rightarrow b_1 = 0$$

$$\Rightarrow T_n = b_n \cos\left(\frac{an\pi}{L} t\right)$$

Now Put it together

$$u_n = T_n X_n = C_n \sin\left(\frac{n\pi}{L} x\right) b_n \cos\left(\frac{an\pi}{L} t\right)$$

combine constants

$$u_n = T_n X_n = C_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{an\pi}{L} t\right)$$

$$\Rightarrow u = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{an\pi}{L} t\right)$$

Apply Final Initial Condition

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) = \frac{1}{4} x (L - x)$$

This is a sine series on interval $[0, L]$

$\therefore\ a_n = \frac{2}{L}\int_0^L \frac{1}{4}x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \frac{1}{2L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{L^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$\therefore\ \boxed{u(x,0) = \frac{L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1-(-1)^n\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi t}{L}\right)}$$

#2 Many steps are the same as #1

Let $u = XT$

This produced to differential equations

$$X'' = -\lambda X \Rightarrow X = c_1\sin(\lambda^{1/2}x) + c_2\cos(\lambda^{1/2}x)$$
$$T'' = -\lambda a^2 T \Rightarrow T = b_1\sin(a\lambda^{1/2}t) + b_2\cos(a\lambda^{1/2}t)$$

Apply boundary conditions yielded

$$X_n = C_n\sin(\lambda^{1/2}x) \quad \text{when } \lambda^{1/2} = \frac{n\pi}{L}$$

$$T = b_1\sin(a\lambda^{1/2}t) + b_2\cos(\lambda^{1/2}t)$$

Now the problem ~~bee~~ goes in different direction

Apply $u(x,0)=0$ condition

$$T = b_1(0) + b_2(1) = 0 \Rightarrow b_2 = 0$$

$$\therefore T_n = b_n \sin\left(\frac{an\pi}{L} t\right) = b_n \sin\left(\frac{an\pi}{L} t\right)$$

$$\therefore u_n = \underbrace{a_n \sin\left(\frac{n\pi}{L}x\right) b_n}_{\text{combine } = c_n} \sin\left(\frac{an\pi}{L}t\right)$$

$$\Rightarrow u_n = c_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{an\pi}{L}t\right)$$

$$\Rightarrow u = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{an\pi}{L}t\right)$$

$$\Rightarrow \frac{du}{dt} = \sum_{n=1}^{\infty} \frac{c_n a n\pi}{L} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right)$$

$$= \frac{a\pi}{L} \sum_{n=1}^{\infty} \underbrace{n c_n}_{\text{combine } = c_n} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{an\pi}{L}t\right)$$

$$\left.\frac{du}{dt}\right|_{t=0} = \frac{a\pi}{L} \sum_{n=1}^{\infty} n c_n \sin\left(\frac{n\pi}{L}x\right) = x(L-x)$$

this is a sine series

$$\Rightarrow nC_n = \frac{2}{L}\int_0^L (x)(L-x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{-4L^3}{n^3\pi^3}(-1)^n + \frac{4L^2}{n^3\pi^3} = \frac{4L^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$\therefore \boxed{u = \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1-(-1)^n\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{an\pi}{L}t\right)}$$