Homework Solutions Section 6.1 PS Solution

#17, 27

17) Let $y = \sum_{n=0}^{\infty} C_n x^n \Rightarrow xy = \sum_{n=0}^{\infty} C_n x^{n+1}$

$y' = \sum_{n=1}^{\infty} C_n (n) x^{n-1}$

$y'' = \sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2} \Rightarrow$

$\Rightarrow y'' + xy = \sum_{n=2}^{\infty} C_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} C_n x^{n+1} = 0$

Indice Shift $\begin{cases} k = n-2 & k = n+1 \\ n = k+2 & n = k-1 \\ n=2 \Rightarrow k=0 & n=0 \Rightarrow k=1 \end{cases}$

$\Rightarrow \sum_{k=0}^{\infty} C_{k+2} (k+2)(k+1) x^k - \sum_{k=1}^{\infty} C_{k-1} x^k = 0$

peel off '0'-term

$\Rightarrow C_2 (2)(1) x^0 + \sum_{k=1}^{\infty} C_{k+2} (k+2)(k+1) x^k - \sum_{k=1}^{\infty} C_{k-1} x^k = 0$

combine

$\Rightarrow C_2 + \sum_{n=1}^{\infty} \left[C_{k+2}(k+2)(k+1) - C_{k-1} \right] x^k = 0$

$\Rightarrow 2C_2 = 0 \Rightarrow C_2 = 0$

$C_{k+2}(k+2)(k+1) - C_{k-1} = 0 \Rightarrow C_{k+2} = \frac{+C_{k-1}}{(k+1)(k+2)}$

$k=1 \Rightarrow C_3 = \frac{+C_0}{(2)(3)}$ $k=2 \Rightarrow C_4 = \frac{C_1}{(3)(4)}$

$k=3 \Rightarrow C_5 = \frac{C_2}{(4)(5)} = 0 \Rightarrow C_8, C_{11}, C_{14}, \ldots = 0$

Find the Patterns is any

$$\begin{cases} k=1 \Rightarrow C_3 = \dfrac{C_0}{(2)(3)} & k=2 \Rightarrow C_4 = \dfrac{C_1}{(3)(4)} \\ k=4 \Rightarrow C_6 = \dfrac{C_3}{(5)(6)} = \dfrac{C_0}{(2)(3)(5)(6)} & k=5 \Rightarrow C_7 = \dfrac{C_4}{(6)(7)} = \dfrac{C_1}{(3)(4)(6)(7)} \end{cases}$$

$\vdots$

$$\therefore \begin{cases} y_1 = C_0 \Big[\underset{C_0}{1} + \underset{C_3}{\dfrac{1}{(2)(3)}} x^3 + \underset{C_6}{\dfrac{1}{(2)(3)(5)(6)}} x^6 + \cdots \Big] \\ y_2 = C_1 \Big[\underset{C_1}{x} + \underset{C_4}{\dfrac{1}{(3)(4)}} x^4 + \underset{C_7}{\dfrac{1}{(3)(4)(6)(7)}} x^7 \cdots \Big] \end{cases}$$

$$\Rightarrow \boxed{y = C_0 y_1 + C_1 y_2}$$

27) $(x^2+2)\sum_{n=2}^{\infty} c_n(n)(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} c_n(n)x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$

$\uparrow y''$ $\qquad \uparrow y'$ $\qquad \uparrow y$

$= \sum_{n=2}^{\infty} c_n(n)(n-1)x^n + \sum_{n=2}^{\infty} 2c_n(n)(n-1)x^{n-2} + \sum_{n=1}^{\infty} 3c_n(n)x^n - \sum_{n=0}^{\infty} c_n x^n = 0$

$n=k$ $\qquad$ Indice Shift $\begin{cases} k=n-2 \\ n=k+2 \\ n=2 \Rightarrow k=0 \end{cases}$ $\qquad n=k \qquad n=k$

$\sum_{k=2}^{\infty} c_k(k)(k-1)x^k + \sum_{k=0}^{\infty} 2c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} 3c_k(k)x^k - \sum_{n=0}^{\infty} c_k x^k = 0$

"peel off $k=0,1$" $\qquad$ "peel off $k=1$" $\qquad$ "peel off $k=0,1$"

$\sum_{k=2}^{\infty} c_k(k)(k-1)x^k + 2c_2(2)(1)x^0 + 2c_3(3)(2)x + \sum_{k=2}^{\infty} 2c_{k+2}(k+2)(k+1)x^k$

$+ 3c_1(1)x + \sum_{k=2}^{\infty} 3c_k(k)x^k - c_0 - c_1 x - \sum_{k=2}^{\infty} c_k x^k = 0$

$\Rightarrow (4c_2 - c_0) + (12c_3 + 2c_1)x + \sum_{k=2}^{\infty}\left[\overbrace{c_k(k)(k-1)}^{k^2-k} + 2c_{k+2}(k+2)(k+1) + 3c_k(k) - c_k\right]x^k$

$k=0 \qquad k=1$

$\Rightarrow 4c_2 - c_0 = 0, \quad 12c_3 + 2c_1 = 0 \qquad \left((k^2+2k-1)c_k + 2c_{k+2}(k+2)(k+1)\right) = 0$

$\Rightarrow c_2 = \frac{1}{4}c_0 \qquad c_3 = \frac{-1}{6}c_1, \qquad c_{k+2} = \frac{-(k^2+2k-1)}{(2)(k+2)(k+1)}$

$\Rightarrow c_4 = \frac{-7}{(3)(4)}c_2 = \frac{-7}{(2)(3)(4)}\left(\frac{1}{4}\right)c_0 = \frac{-7}{4\,4!}c_0$

$\Rightarrow c_5 = \frac{-(14)}{(2)(5)(4)}c_3 = \frac{-(14)}{(2)(5)(4)}\left(\frac{-1}{6}c_1\right) = \frac{14}{(2)(5!)}c_1$

2×3

$$C_6 = \frac{-(23)}{(2)(6)(5)} C_4 = \frac{(-27)(-7)}{(2)(4)(4! \times 5 \times 6)} C_0 = \frac{(7)(23)}{8(6!)} C_0$$

$$C_7 = \frac{-(34)}{(2)(7)(6)} C_5 = \frac{(-34)(14)}{(4)(5! \times 6 \times 7)} = \frac{-(14)(34)}{(4)(7!)}$$

$$\therefore\ y_1 = C_0\left[1 + \frac{1}{2!}x^2 - \frac{7}{4 \cdot 4!}x^4 + \frac{(7)(23)}{(8)(6!)}x^6 \cdots\right]$$

$$y_2 = C_1\left[x - \frac{1}{3!}x^3 + \frac{7}{5!}x^5 - \frac{(7)(17)}{7!}x^7 \cdots\right]$$

ARE you Smart Enough to spot a pattern, I'm not ☺