Supplimentary Lecture 2
Homework Solutions

1. Express $f(x)$ as a power series centered on a

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

a) Thus $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x+a)^3 \ldots$

$$\Rightarrow f(a) = c_0$$

b) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$

$$\Rightarrow f'(a) = c_1$$

c) $f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 \ldots$

$$\Rightarrow f''(a) = 2c_2$$

d) $f'''(x) = 6c_3(x-a) + 24c_4(x-a) \ldots$

$$\Rightarrow f''(a) = 6c_3 \quad \ldots \text{ etc.}$$

e) therefore:

$$\left.\begin{aligned}
c_0 &= f(a) = \frac{f^{(0)}(a)}{0!} \\
c_1 &= f'(a) = \frac{f^{(1)}(a)}{1!} \\
c_2 &= \frac{f''(a)}{2} = \frac{f^{(2)}(a)}{2!} \\
c_3 &= \frac{f'''(a)}{6} = \frac{f^{(3)}(a)}{3!}
\end{aligned}\right\}$$

Therefore By Deduction

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

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2)

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ |
| --- | --- | --- |
| 0 | $e^x$ | 1 |
| 1 | $e^x$ | 1 |
| 2 | $e^x$ | 1 |
| 3 | $e^x$ | 1 |
| 4 | $e^x$ | 1 |

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

3) a) $$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} \dots}$$

b) $$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \dots}$$

c) $$-2xe^{-x^2} = \frac{d}{dx}\left(e^{-x^2}\right) = \sum_{n=1}^{\infty} (-1)^n (2n) \frac{x^{2n-1}}{n!}$$

$$= \boxed{2\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(n-1)!} = -2x + 2x^3 - x^5 + \frac{x^7}{3} - \frac{x^9}{12} \dots}$$