Lecture Notes

SECTION 1.2
Init Val. Prob.

I FIRST ORDER IVP

solve $\frac{df}{dx} = f(x,y)$ subject to $y(x_0) = y_0$

Example: $y' = y$ or $\frac{dy}{dx} = y$ where $y(0) = 3$

Solution $\Rightarrow \boxed{y = ce^x} \Rightarrow y(0) = ce^0 = C = 3$

Apply initial condition

$\Rightarrow \boxed{y = 3e^x}$ "family of solutions"

II Second Order IVP

$\frac{d^2y}{dx^2} = f(x,y)$ subject to $\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$

Example: $x'' + 4x = 0$ or $\frac{d^2x}{dt^2} + 2x = 0$

subject to $\begin{cases} x(\pi/2) = -2 \\ x'(\pi/2) = 1 \end{cases}$

Solution: $y = C_1\sin(2x) + C_2\cos(2x)$ ⟸ "family of soln's"

Apply 1st Initial Condition $\longrightarrow y(\frac{\pi}{2}) = C_1 \underbrace{\cancel{\sin(\pi)}}_{0} + C_2 \underbrace{\cancel{\cos(\pi)}}_{-1} = -2 \Rightarrow C_2 = 2$

$\Rightarrow y = C_1\sin(2x) + 2\cos(2x)$

Apply 2nd Initial Condition $\longrightarrow y' = 2C_1\cos(2x) - 4\sin(2x) = 1$

$\Rightarrow y'(\pi/2) = 2C_1 \underbrace{\cancel{\cos(\pi)}}_{-1} - 4 \underbrace{\cancel{\sin(\pi)}}_{0} \Rightarrow C = -1/2$

$\Rightarrow \boxed{y = -\frac{1}{2}\sin(2x) + 2\cos(2x)}$

III) A D.E. WITH MORE THAN ONE SOLUTION

$\frac{dy}{dx} = xy^{1/2}$ subject to $y(0)=0$

$\Rightarrow$ Both $y=0$ and $y=\frac{1}{16}x^4$ work!!

This is not good!!
How can I gaurantee that my solution is unique..

IV Theorem: Given: $\frac{dy}{dx} = f(x,y)$ subject to $y(x_0)=y_0$

If $\frac{\partial f}{\partial y}(x_0,y_0)$ is continuous

$\Rightarrow$ Then a unique solution exists at (x_0,y_0)

Note: THEOREM APPLIES TO 1ST ORDER EQUATIONS ONLY!

NOTE If $\frac{\partial f}{\partial y}(x_0,y_0)$ not continuous $\Rightarrow$ Then no conclusion!

Example: Can we be sure that $\frac{dy}{dx} = xy^{1/2}$ has a unique solution that passes through $(0,0)$

$f(x,y) = xy^{1/2}$ $\qquad \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$

$\Rightarrow \frac{\partial f}{\partial y}(0,0) = \frac{0}{0} \Rightarrow$ undefined $\Rightarrow$ not continuous

$\therefore$ unique solution not gauranteed!

$\therefore$ for $y \neq 0 \Rightarrow$ unique soln is certain