Lecture Notes

Section 2.2
Separation of Variables

I. Separation of Variables

Recall 1^{ST} ORDER D.E. $\frac{dy}{dx} = f(x,y)$

If $f(x,y)$ can be separated, i.e. $f(x,y) = g(x)h(y)$
Then we can solve DE using separation of var

A Example:

$$\frac{dy}{dx} = -\frac{x}{y} = (-x)\left(\frac{1}{y}\right)$$

$\Rightarrow \int y\,dy = \int -x\,dx$ solve by integration

$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$

attach constant of integration to independent variable

$\Rightarrow \frac{1}{2}y^2 + \frac{1}{2}x^2 = C \Rightarrow x^2 + y^2 = C$

you don't necesserily have to solve for 'y'. This is an implicit solution.

B. Example

$$\frac{dy}{dx} = y^2 - 1 \Rightarrow \frac{dy}{(y^2-1)} = dx$$

$$\Rightarrow \int \frac{1}{(y+1)(y-1)} dy = \int dx$$

partial fraction method $\Rightarrow \frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$

$$\Rightarrow 1 = (y-1)A + (y+1)B$$

$$\Rightarrow y=1 \Rightarrow B = 1/2 \quad \Rightarrow y=-1 \Rightarrow A = -1/2$$

$$\int \left(-\frac{1}{2}\frac{1}{y+1} + \frac{1}{2}\frac{1}{y-1}\right) dy = x + C \Rightarrow \frac{1}{2}\int \frac{1}{y-1} - \frac{1}{y+1}\, dy = x + C$$

$$\Rightarrow \ln|y-1| - \ln|y+1| = 2x + C \Rightarrow \ln\left|\frac{y-1}{y+1}\right| = 2x + C$$

$$\Rightarrow \frac{y-1}{y+1} = e^{2x+C} = e^{2x}e^{C} = ke^{2x}$$

$$\Rightarrow y - 1 = kye^{2x} + ke^{2x} \Rightarrow y - kye^{2x} = ke^{2x} + 1$$

$$\Rightarrow y(1-ke^{2x}) = ke^{2x} + 1 \Rightarrow \boxed{y = \frac{ke^{2x}+1}{1-ke^{2x}}}$$

Note: We sometimes lose a solution w/this method. In this case $y=-1$ is lost.