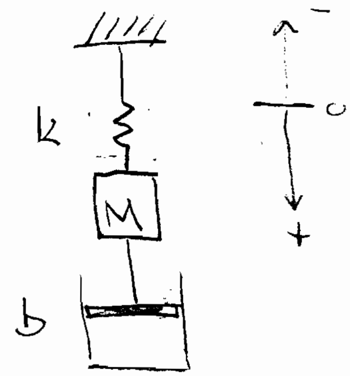
Lecture Notes

Section 5.1.2
mass/Spring System
FREE Damped Motion

I Schematic/EQUATION

k = spring constant
m = mass
b = damping coefficient



$$mx'' + bx' + kx = 0$$

damping factor resists velocity

II Solutions

$$(mD^2 + bD + k)x = 0$$

$$D = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

a) $b = 0 \Rightarrow$ undamped motion (imaginary root $\pm Bi$)

b) $b^2 - 4mk > 0 \Rightarrow$ overdamped motion (2 real roots r_1, r_2)

c) $b^2 - 4mk = 0 \Rightarrow$ critically damped (2 repeated roots 'r')

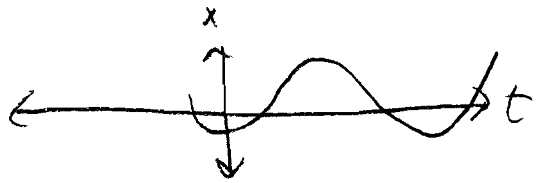
d) $b^2 - 4mk \leq 0$ underdamped (complex roots $\alpha \pm Bi$)

Note: since $m, b, k > 0$ then r_1, r_2, r, α above are all less than 0

III Characteristics of Solutions

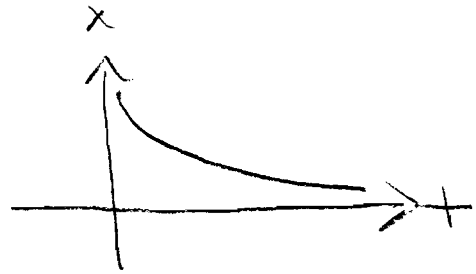
a) undamped

$$x = c_1 \sin(Bx) + c_2 \cos(Bx)$$
$$= A \sin(Bx + \phi)$$



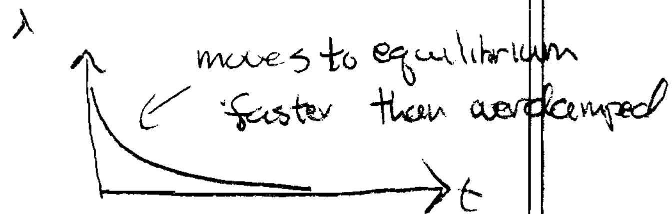
b) overdamped

$$x = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
$$r_1, r_2 < 0$$



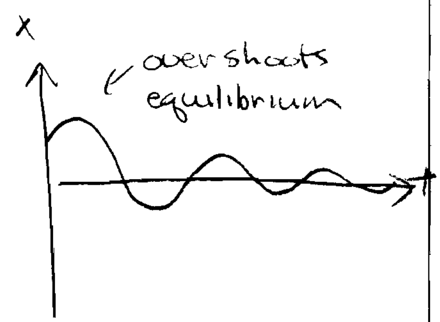
c) critically damped

$$x = c_1 e^{rx} + c_2 x e^{rx}$$
$$r < 0$$



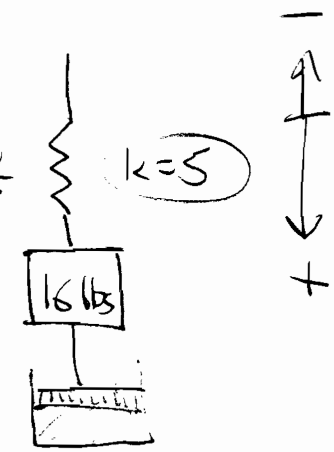
d) underdamped

$$x = c_1 e^{\alpha x} \sin(Bx) + c_2 e^{\alpha x} \cos(Bx)$$
$$= e^{\alpha x}(c_1 \sin(Bx) + c_2 \cos(Bx))$$
$$= A e^{\alpha x} \sin(Bx + \phi)$$
$$\alpha < 0$$



IV Example 5 (p 188)

16 lbs stretches spring (3.2-5) ft   find k $k=5$



$\therefore F = kx \Rightarrow 16 = 3.2k \Rightarrow k=5$

$\Rightarrow mx'' + bx' + kx = 0$ $b=1$

$m = W/g = 16/32 = .5$

$\therefore .5x'' + x' + 5x = 0 \Rightarrow x'' + 2x' + 10x = 0$

$\Rightarrow (D^2 + 2D + 10)x = 0$

$\Rightarrow D = \dfrac{-2 \pm \sqrt{4-40}}{2} = \dfrac{-2 \pm \sqrt{-36}}{2} = \dfrac{-2 \pm 6i}{2} = 1 \pm 3i$

$\Rightarrow x = c_1 e^{-t}\sin(3t) + c_2 e^{-t}\cos(3t)$

initial conditions: $\begin{cases} x(0) = -2 \\ x'(0) = 0 \end{cases}$

$x(0) = c_1 e^0 \underbrace{\cancel{\sin(0)}}_{\Rightarrow 0} + c_2 e^0 \underbrace{\cos(0)}_{\Rightarrow 1} = -2$

$x = c_1 e^{-t}\sin(3t) - 2e^{-t}\cos(3t)$

$x' = -c_1 e^{-t}\sin(3t) + 3c_2 e^{-t}\cos(3t)$

$\quad + 2e^{-t}\cos(3t) + 6e^{-t}\sin(3t)$

$\Rightarrow x'(0) = 3c_2 + 2 = 0 \Rightarrow c_2 = -3/2$

$\therefore x = e^{-t}\left(-\frac{2}{3}\sin(3t) - 2\cos(3t)\right)$

put in combined form

$$A = \left(2^2 + \left(\tfrac{2}{3}\right)^2\right)^{1/2} = \left(4 + \tfrac{4}{9}\right)^{1/2} = \left(\tfrac{40}{9}\right)^{1/2} = \frac{2\sqrt{10}}{3}$$

$$\phi = \tan^{-1}\left(\frac{C_{cos}}{C_{sin}}\right) = \tan^{-1}\left(\frac{-2}{-2/3}\right) = \tan^{-1}(3)$$

$$= 1.249 \quad +\pi = 4.391$$

since $C_{sin} < 0$

$$\Rightarrow \boxed{x = \frac{2\sqrt{10}}{3} e^{-t} \sin(3t + 4.391)}$$