Lecture Notes

Appendix II.4
Eigenvalues / Eigenvectors

I The Eigenvalue PROBLEM

Let: ① A be an $n \times n$ matrix
② $\vec{K}$ be a non-zero vector

Then: λ is said to be a **eigenvalue** if:

$$A\vec{K} = \lambda \vec{K}$$

Also: $\vec{K}$ is said to be the eigenvector corresponding to the eigenvalue λ.

Example

Let $A = \begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix}$ $\vec{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{K}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

$\Rightarrow A K_1 = \begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e. $A\vec{K}_1 = \underline{1}\, \vec{K}_1$

$\therefore$ $\boxed{\lambda = 1, \ \vec{K}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ are an eigenvalue/eigenvector pair

$\Rightarrow A K_2 = \begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 12 \\ 42 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

$\Rightarrow$ $\boxed{\lambda = 6, \ \vec{K}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}}$ are a "pair"

$\Rightarrow$ now try $k_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

i.e. not a multiple of $\vec{k}_2$. Therefore not an eigenvalue eigenvector pair

II <u>How to Find Eigenvalues / Eigenvectors</u>

Example: Let $A = \begin{bmatrix} -1 & 2 \\ -7 & 8 \end{bmatrix}$

① Set up matrix $\begin{bmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{bmatrix}$

<u>note</u>: λ is subtracted from diagonal elements

② Find λ for $\det\left(\begin{bmatrix} -1-\lambda & 2 \\ -7 & 8-\lambda \end{bmatrix}\right) = 0$

determinant

$$\Rightarrow (-1-\lambda)(8-\lambda) - (2)(-7) = 0$$

$$\Rightarrow -8 + \lambda - 8\lambda + \lambda^2 + 14 = 0 \Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 6) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 1, \ \lambda_2 = 6}$$

③ <u>Now Find EIGENVECTOR FOR $\lambda = 1$</u>

$\Rightarrow$ set up following

$$\begin{bmatrix} -1-\lambda_1 & 2 \\ -7 & 8-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1-1 & 2 \\ -7 & 8-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ -7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -2x_1 + 2x_2 = 0 \\ -7x_1 + 7x_2 = 0 \end{matrix}$$

$\Rightarrow$ only need to use 1 equation to find vector since they are basically the same equation

$$\Rightarrow \quad -2x_1 + 2x_2 = 0 \Rightarrow \quad x_1 = x_2$$

$\Rightarrow$ pick any value for x_1, but pick a simple value, i.e. $x_1 = 1$

$\Rightarrow \therefore \; x_2 = 1$

$$\therefore \boxed{\vec{k}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

④ Eigenvector for $\lambda = 6$

$$\begin{bmatrix} -1-6 & 2 \\ -7 & 8-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow -7x_1 + 2x_2 = 0 \Rightarrow 2x_2 = 7x_1$$

$$\Rightarrow x_2 = \frac{7}{2} x_1$$

⇒ pick any value for x_1, but <u>keep it simple!!</u>

i.e. $x_1 = 2 \Rightarrow x_2 = 7$

$$\therefore \boxed{\vec{K}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}}$$