Maclaurin Series

In the early 1700's a Scottish mathematician, Colin Maclaurin, made use of a series that stated that any function could be expressed as a power series expanded about 0, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \frac{f^{(n)}(0)}{n!}.$$

Note that $f^{(n)}(0)$ is the n^th^-derivative of $f(x)$ evaluated at 0. In expanded form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \dots$$

Proof of Maclaurin Series:

- Start with: $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$
- Therefore: $f(0) = c_0$ or $c_0 = \frac{f^{(0)}(0)}{0!}$ (note: 0!=1)
- Now consider: $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$
- Therefore: $f'(0) = c_1$ or $c_1 = \frac{f'(0)}{1!}$
- Now consider: $f''(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$
- Therefore: $f''(0) = 2c_2$ or $c_2 = \frac{f''(0)}{2!}$
- Now consider: $f'''(x) = 6c_3 + 24c_4 x + \cdots$
- Therefore: $f'''(0) = 6c_3$ or $c_3 = \frac{f'''(0)}{3!}$
- Etc …
- We can ***imformally*** deduce that $c_n = \frac{f^{(n)}(0)}{n!}$, hence we have derived the Maclaurin series.

Example: Maclaurin Series for sin(x):

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n
0	$\sin(x)$	0	0
1	$\cos(x)$	1	$1/1!$
2	$-\sin(x)$	0	0
3	$-\cos(x)$	-1	$-1/3!$
4	$\sin(x)$	0	0
5	$\cos(x)$	1	$1/5!$
6	$-\sin(x)$	0	0
7	$-\cos(x)$	-1	$1/7!$

Therefore:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example: Radius of Convergence for Maclauring Seris of sin(x)

Perform a Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| < 1 \rightarrow$$

You can ignore -1's since will become +1's:

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\frac{(2n+1)!}{(2n+3)!}\right| < 1 \rightarrow \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| < 1$$

Pulling out the x^2 expression (note that it is always positive so absolute value is dropped):

$$x^2\lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+3)}\right| < 1$$

Since the limit goes to 0, the expression is true for all x, i.e. the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

Computer Demo (goB)

Using the Series for sin(x) to Find a Series for sin(x^2).

Simply replace all *x*'s with x^2's, i.e:

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}$$

Using the Series for sin(x) to Find a Series for cos(x).

Since the derivative of sin(x) is cos(x), simply take the derivative of the sin(x) series:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Taylor Series

Brook Taylor proposed the following series before Maclaurin:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ where } c_n = \frac{f^{(n)}(a)}{n!}.$$

Note that $f^{(n)}(a)$ is the nth-derivative of $f(x)$ evaluated at a. In expanded form the Taylor series becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 \ldots$$

This defines $f(x)$ as a power series centered on *a.* It is a more general statement then the Maclaurin series (note: if a=0, the Tayler series is the Maclaurin series). It is also slightly more complicated to calculate the Taylor series for a function.

Example: Taylor Series for sin(x) Centered on $\pi/2$:

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$	c_n
0	$\sin(x)$	1	$1/0!$
1	$\cos(x)$	0	0
2	$-\sin(x)$	-1	$-1/2!$
3	$-\cos(x)$	0	0
4	$\sin(x)$	1	$1/4!$
5	$\cos(x)$	0	0
6	$-\sin(x)$	-1	$-1/6!$

Therefore:

$$\sin(x) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 \ldots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$$

Homework:

1. Derive the Taylor series for a function *f(x)* centered on *a*. (Hint: Follow the derivation of the Maclaurin series).
2. Derive a Maclaurin series for e^x.
3. Use your result to find Maclaurin series for:
   a. e^{x^2}
   b. e^{-x^2}
   c. $-2xe^{-x^2}$