Lecture Notes

Section 11.1
Orthogonal Functions

I Inner Product (u, v)

A. For vectors the inner product is the dot product

$$\therefore (\vec{u}, \vec{v}) = \vec{u} \cdot \vec{v}$$

i.e. $\vec{u} = \langle 1, 2, 3 \rangle$ $\vec{v} = \langle 4, 0, 1 \rangle$

$$\Rightarrow (\vec{u}, \vec{v}) = \langle 1, 2, 3 \rangle \cdot \langle 4, 0, 1 \rangle$$
$$= (1)(4) + (2)(0) + (3)(1) = 7$$

B For Functions: Inner Product on the Interval $[a, b]$ is defined as

$$\boxed{(f(x), g(x)) = \int_a^b f(x) g(x)\, dx}$$

i.e. $f(x) = x$ $\quad g(x) = x^2$ interval is $[-1, 2]$

inner product on $[-1, 2]$ = $\int_{-1}^{2} (x)(x^2)\, dx$

$$= \int_{-1}^{2} x^3\, dx = \frac{1}{4} x^4 \Big|_{-1}^{2} = \frac{16}{4} - \frac{1}{4} = \boxed{\frac{15}{4}} \checkmark$$

II Orthogonality

A For Vectors

$\vec{u}$ & $\vec{v}$ are orthogonal if

$$(\vec{u},\vec{v}) = \vec{u}\cdot\vec{v} = 0$$

i.e. $\langle 1,-2,1\rangle$ and $\langle 4,1,-2\rangle$ are orthogonal

B For Functions

$f(x)$ and $g(x)$ are orthogonal on interval $[a,b]$ if

$$\boxed{\int_a^b f(x)g(x) = 0}$$

i.e. $f(x) = x$ $\quad g(x) = x^2$ $\quad$ interval $= [-1,1]$

$$\Rightarrow \int_{-1}^{1} (x)(x^2)dx = \frac{1}{4}x^4 \Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0 \checkmark\checkmark$$

III Norm

A. For Vectors: The norm of $\vec{u}$

$$\|\vec{u}\|^2 = (\vec{u}, \vec{u}) = \vec{u} \cdot \vec{u}$$

symbol for norm of $\vec{u}$

inner product of $\vec{u}$ with itself

$\therefore \vec{u} = \langle 1,2,3 \rangle \Rightarrow \|\vec{u}\|^2 = \langle 1,2,3 \rangle \cdot \langle 1,2,3 \rangle$

$= 1+4+9 = 14$

B. For Functions: norm on interval $[a,b]$

$$\|f(x)\|^2 = \int_a^b [f(x)]^2 \, dx$$

i.e $f(x) = x$ on interval $[-2,3]$

$$= \|f(x)\|^2 = \int_{-2}^{3} x^2 \, dx = \frac{1}{3} x^3 \Big|_{-2}^{3}$$

$$= \frac{1}{3}(27-(-8)) = \frac{35}{3} \checkmark$$

D. Orthogonal Set of Functions

A set of functions $\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots\}$ is orthogonal on $[a,b]$ if

$$(\phi_m(x), \phi_n(x)) = \boxed{\int_a^b \phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n}$$

E. Orthonormal Set of Functions

A set of functions is said to be orthonormal on $[a,b]$ if

$$\boxed{\int_a^b \phi_m(x)\phi_n(x)\,dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}}$$

Example: Orthogonal Functions (Ex 1, 399)