Lecture Notes

Section 11.2
Fourier Series

I ORTHOGONAL Series EXPANSION

(1) Assume that $\{\phi_0(x), \phi_1(x), \ldots \phi_n(x)\}$ is an orthogonal set of functions on $[a,b]$

(2) And $f(x)$ is a function defined on $[a,b]$

(3) Then we can rewrite $f(x)$

i.e. $\boxed{f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) \ldots + c_n\phi_n(x)}$

(4) Solving for C_n

(a) Say I want to solve for C_2

(b) multiply equation by $\phi_2(x)$ & integrate from $[a,b]$

$$\int_a^b f(x)\phi_2(x)\,dx = c_0\int_a^b \phi_0(x)\phi_2(x)\,dx + c_1\int_a^b \phi_1(x)\phi_2(x)\,dx$$
$$+ c_2\int_a^b \phi_2(x)\phi_2(x)\,dx \ldots + c_n\int_a^b \phi_n(x)\phi_2(x)\,dx$$

(c) Recall Definition of Orthogonality

$$\boxed{\int_a^b \phi_n(x)\phi_m(x)\,dx \begin{cases} = 0 & m \neq n \\ \neq 0 & m = n \end{cases}}$$

(d) Therefore Equation (1) becomes

$$\int_a^b f(x)\,\phi_2(x)\,dx = c_2 \int_a^b [\phi_2(x)]^2\,dx$$

$$\Rightarrow c_2 = \frac{\int_a^b f(x)\,\phi_2(x)\,dx}{\int_a^b [\phi_2(x)]^2\,dx}$$

(e) OR in General

$$\boxed{c_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\int_a^b [\phi_n(x)]^2\,dx}}$$

II Fourier Series

a) Consider the orthogonal set of functions

$$\left\{1,\ \sin\frac{n\pi x}{p},\ \cos\frac{n\pi x}{p}\right\} \text{ on interval } [-p, p]$$

NOTE

$$\boxed{\begin{cases}\int_{-p}^{p} \sin\frac{n\pi x}{p}\,dx = 0\\ \int_{-p}^{p} \cos\frac{n\pi x}{p}\,dx = 0\end{cases}}$$

$$\boxed{\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)dx = 0}$$

$$\boxed{\int_{-p}^{p} dx = 2p}$$

$$\boxed{\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right)\sin\left(\frac{m\pi x}{p}\right)dx = \begin{cases} 0 & m \neq n \\ p & m = n\end{cases}}$$

$$\boxed{\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\cos\left(\frac{m\pi x}{p}\right)dx = \begin{cases} 0 & m \neq n \\ p & m = n\end{cases}}$$

b) Therefore $\{1, \cos\frac{n\pi x}{p}, \sin\frac{n\pi x}{p}\}$ is an orthogonal set of functions on the interval $[-p, p]$

c) This implies that a function $f(x)$ can be rewritten as:

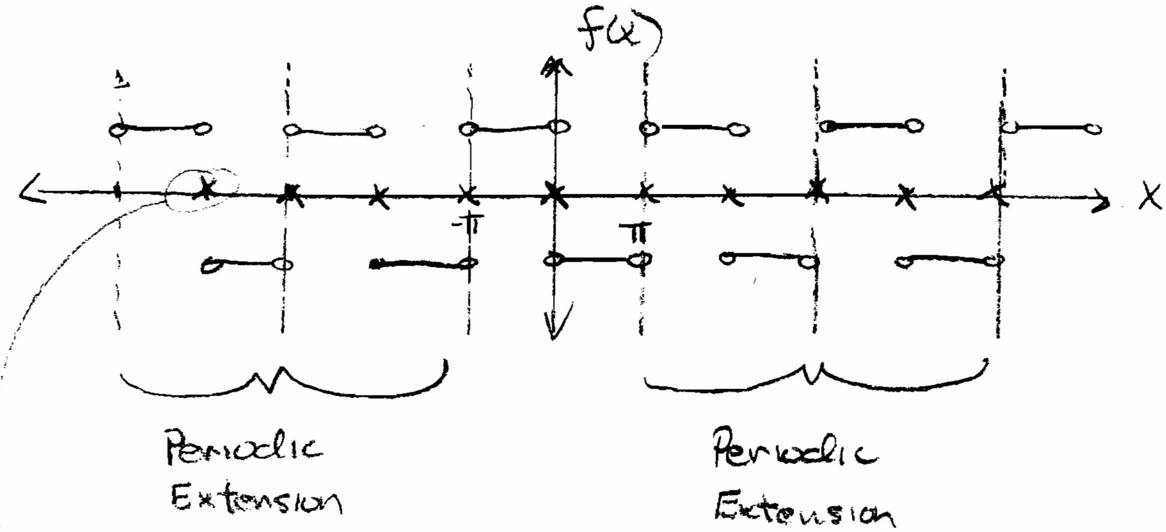
$$\boxed{\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{p}\right) + b_n\sin\left(\frac{n\pi x}{p}\right) \\ &\text{where} \\ a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx \end{aligned}}$$

III. <u>Example</u>: Problem 1, p. 407 +

Maple Demo

III Plot of FS After INFINITE Terms

consider $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ -1 & 0 < x < \pi \end{cases}$



At jump discontinuities the FS plot takes on values ½ way between the "jump"

IV "Fun & Games" – Problem 17 (407)

Show $\pi^2/6 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$ using result in Problem 5

For $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$ $\boxed{\text{Hint: } f(\pi) = \pi^2/2}$

$$f(x) \sim \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2}\cos(nx) + \left(\frac{-\pi}{n}(-1)^n + \frac{2}{n^3\pi}((-1)^n - 1) \right)\sin(nx) \right]$$