Lecture Notes

Section 12.1
Solving PDE's by Separation of Variables

I Recall Separation of Variables For DEs

i.e. $\frac{dy}{dx} = x^3 y \qquad y(0) = e^2$

$\Rightarrow \frac{dy}{y} = x^3 dx$ "separate"

$\Rightarrow \int \frac{dy}{y} = \int x^3 dx$ "integrate"

$\Rightarrow \ln|y| = \frac{1}{4}x^4 + c \Rightarrow y = e^{\frac{1}{4}x^4 + c}$

$\Rightarrow y(0) = e^{\frac{1}{4}(0) + c} = e^2 \Rightarrow c = 2$ "consummate" i.e. apply initial condition

$\Rightarrow \boxed{y = e^{\frac{1}{4}x^4 + 2}}$ "celebrate" i.e. the answer

II Separation Methods Applied to 1st ORDER PDEs

i.e. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \quad u(x,0) = 3e^{-2x}$

note: that u is a function of 'x' and 'y'

i.e. $u = f(x,y)$

A. Separate

① To Separate, Assume that

$$u = X(x)\,Y(y)$$

i.e. the function $f(x,y)$ can be separated into a "function of x" times a "function of y"

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[X(x)Y(y)\right] = \underbrace{\frac{\partial X}{\partial x}Y + X\cancel{\frac{\partial Y}{\partial x}}^{\,0}}_{\text{Product Rule}} = \boxed{X'Y}$$

Note: $\frac{\partial Y}{\partial x} = 0$ because Y is a function of 'y' and not a function of 'x'

$$\frac{\partial u}{\partial y} = \cancel{\frac{\partial X}{\partial y}}^{\,0} Y + X\frac{\partial Y}{\partial y} = \boxed{XY'}$$

② we can now rewrite the PDE

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \Rightarrow X'Y = XY'$$

③ Divide PDE by XY to separate variables

$$\frac{X'Y}{XY} = \frac{XY'}{XY} \Rightarrow \frac{X'}{X} = \frac{Y'}{Y} = -\lambda$$

The only way then can be equal is if the are a constant $\to \lambda$

this is function of 'x' only

this is a function of 'y' only

$\Rightarrow$ Note: I choose '$-\lambda$' vice 'λ' for convenience

4) We Now have turned the PDE into 2 DEs.

$$\frac{X'}{X} = -\lambda \Rightarrow X' = -\lambda X \Rightarrow \boxed{X' + \lambda X = 0}$$

$$\frac{Y'}{Y} = -\lambda \Rightarrow Y' = -\lambda Y \Rightarrow \boxed{Y' + \lambda X = 0}$$

B. Solve the DE's

a, b are constants

$$\boxed{\begin{array}{l} X = a e^{-\lambda x} \\ Y = b e^{-\lambda y} \end{array}}$$

(recall X is a function of 'x')
(recall Y is a function of 'y')

C.) Put it Together

Since $U = XY \Rightarrow u = \underbrace{ab}_{} e^{-\lambda x} e^{-\lambda y}$

ab is a constant, lets rename it ab = k

$$\Rightarrow \boxed{u = k e^{-\lambda(x+y)}}$$

D) Apply Initial Conditions

$$u(x,0) = k e^{-\lambda x} = 3 e^{-2x} \Rightarrow k = 3, \lambda = -2$$

$$\therefore \boxed{u(x,y) = 3e^{-2(x+y)}}$$

III Separation Methods Applied to 2nd order PDE's

i.e. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \Rightarrow u = f(x,y) = \underbrace{X(x)\,Y(y)}_{\text{assumption}}$

A). Therefore

$$\frac{\partial^2 u}{\partial x^2} = X''Y \qquad \frac{\partial u}{\partial y} = XY'$$

B) Rewrite PDE

$$X''Y = XY'$$

function of 'x' only

function of 'y' only

constant

C) DIVIDE BY XY

$$\frac{X''Y}{XY} = \frac{XY'}{XY} \Rightarrow \frac{X''}{X} = \frac{Y'}{Y} = -\lambda$$

D) WE Now Have 2 DE's

$$\frac{X''}{X} = -\lambda \Rightarrow X'' = -\lambda X \Rightarrow \boxed{X'' + \lambda X = 0}$$

$$\frac{Y'}{Y} = -\lambda \Rightarrow Y' = -\lambda Y \Rightarrow \boxed{Y' + \lambda Y = 0}$$

E) Solve for Y(y)

$$\Rightarrow \boxed{Y = a e^{-\lambda y}}$$

F) Solve for X

$$X'' + \lambda X = 0$$

There are 3 possible solutions for

$\lambda < 0$, $\lambda = 0$, $\lambda > 0$

① $\lambda < 0$

$$X'' - \lambda X = 0 \Rightarrow (D^2 - \lambda) X = 0 \Rightarrow D = \pm \lambda^{1/2}$$

$$\therefore \boxed{X = C_1 e^{\lambda^{1/2} x} + C_2 e^{-\lambda^{1/2} x}}$$

② $\lambda = 0$

$$X'' = 0 \Rightarrow X' = C_1 \Rightarrow \boxed{X = C_1 x + C_2}$$

③ $\lambda > 0$

$$X'' + \lambda X = 0 \Rightarrow (D^2 + \lambda) X = 0 \Rightarrow D = \pm \lambda^{1/2} i$$

$$\therefore \boxed{X = C_1 \sin(\lambda^{1/2} x) + C_2 \cos(\lambda^{1/2} x)}$$

G) Put it all together $u = XY$

– Final Solution depends on if $\lambda < 0, \lambda = 0, \lambda > 0$