SM212 Lecture Notes
Section: 12.3 Heat Equation with $T=0$ Ends

1. Heat Equation

a. PDE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ or $u_t = ku_{xx}$ where $u(x,t)$ is the temperature in the bar.

b. Boundary Conditions: $u(0,t) = u(L,t) = 0$

c. Initial Condition: $u(x,0) = f(x)$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = X(x)T(t) \to u = XT$

a. Therefore $u_t = X\frac{dT}{dt} = XT'$ and $u_{xx} = \frac{d^2X}{dx^2}T = X''T$

b. Rewrite PDE as $u_t = ku_{xx} \to T'X = kTX''$

c. Divide both sides by $kXT \to \frac{XT'}{kXT} = \frac{T}{kXT} \to \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$

- Note: Carry k with T for convenience
- Note: λ must be a constant and is called an eigenvalue
- Note: Used $-\lambda$, again to follow for convenience

3. Now we can generate two differential equations that are solvable.

a. DE in time: $\frac{1}{k}\frac{T'}{T} = -\lambda \to T' = -k\lambda T \to T' + k\lambda T = 0$

b. DE in space: $\frac{X''}{X} = -\lambda \to X'' = -\lambda X \to X'' + \lambda X = 0$

4. Eigenvalue analysis using boundary conditions.

a. Assume $\lambda < 0$ thus: $X'' - \lambda X = 0 \rightarrow X = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

- BC: $X(0) = c_1 + c_2 = 0$ and $X(L) = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L}$
- This implies $c_1 = c_2 = 0 \rightarrow X(x) = 0 \rightarrow \text{trivial}$

b. Assume $\lambda = 0$ thus: $X'' = 0 \rightarrow X = c_1 x + c_2$

- BC: $X(0) = c_2 = 0$ and $X(L) = c_1 L = 0 \rightarrow c_1 = 0$
- This implies $X(x) = 0 \rightarrow \text{trivial}$

c. Assume $\lambda < 0$ thus: $X'' + \lambda X = 0 \rightarrow X = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$

- BC: $X(0) = c_2 = 0$ and $X(L) = c_1 \sin\left(\sqrt{\lambda}L\right) = 0$
- This implies $\sin(\sqrt{\lambda}L) = 0 \rightarrow \sqrt{\lambda}L = n\pi$ where $n = 1,2,3,\ldots$
- This implies $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n = 1,2,3,\ldots$
- By Superposition: $X(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

5. Now solve for $T(t)$.

a. Recall $T' + k\lambda T = 0 \rightarrow T(t) = b_1 e^{-k\lambda t} = b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

6. Now put it all together to get $u(x,t)$

a. $u(x,t) = X(x)T(t) \rightarrow b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \rightarrow \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$

7. Finally Apply Initial Condition

a. $u(x,0)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)= f(x)$

b. This is just a Fourier Sine Series where:

- $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

8. Example

a. PDE: $\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t)=u(1,t)=0$

c. IC: $u(x,0)=(1-x)x^2$

d. General Solution applies with values $L=1$ and $k=5$ plugged in:

$$u(x,t)=\sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)=\boxed{\sum_{n=1}^{\infty} c_n e^{-5(n\pi)^2 t}\sin(n\pi x)}$$

- Note: eigenvalue analysis does not change because boundary conditions were the same.
- What parts of the solution decay the fastest?
- What is the equilibrium solution?

e. Apply initial condition: $u(x,0)=(1-x)x^2$

$$u(x,0)=\sum_{n=1}^{\infty} c_n e^{o}\sin(n\pi x)=\sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

f. Now from Sine series we have:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x)dx = 2\int_0^1 x^2(1-x)\sin(n\pi x)dx$$

g. Plug this into TI:

$$c_n = -8\frac{\cos(n\pi)}{(n\pi)^3} - 2\frac{\sin(n\pi)}{(n\pi)^2} + 12\frac{\sin(n\pi)}{(n\pi)^4} - \frac{4}{(n\pi)^3}$$

h. Recall: $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n \rightarrow$

$$c_n = -8\frac{(-1)^n}{(n\pi)^3} - \frac{4}{(n\pi)^3} = 8\frac{(-1)^{n+1}}{(n\pi)^3} - \frac{4}{(n\pi)^3} = \frac{4}{(n\pi)^3}(2(-1)^{n+1} - 1) \rightarrow$$

i. Final Solution when expanded:

$$\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{2\left(-1\right)^{n+1} - 1}{n^3}e^{-5\left(n\pi\right)^2 t}\sin\left(n\pi x\right) =$$

$$\frac{4}{\pi^3}e^{-5\pi^2 t}\sin\left(\pi x\right) - \frac{3}{2\pi^3}e^{-20\pi^2 t}\sin\left(2\pi x\right) + \frac{4}{27\pi^3}e^{-45\pi^2 t}\sin\left(3\pi x\right) \ldots$$