

SM212 Lecture Notes

Section 12.3 Heat Equation Insulated Ends

1. Heat Equation

- a. PDE: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$
- b. BC: $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$
- c. IC: $u(x, 0) = f(x)$

2. Assume $u(x, t)$ is separable, i.e. $u(x, t) = X(x)T(t) \rightarrow u = XT$

- a. Therefore $u_t = X \frac{dT}{dt} = XT'$ and $u_{xx} = \frac{d^2 X}{dx^2} T = X''T$
- b. Rewrite PDE as $u_t = k u_{xx} \rightarrow XT' = kX''T$
- c. Divide both sides by $kXT \rightarrow \frac{XT'}{kXT} = \frac{kX''T}{kXT} \rightarrow \frac{1}{k} \frac{T'}{T} = \frac{X''}{X} = -\lambda$
- d. DE in time: $\frac{1}{k} \frac{T'}{T} = -\lambda \rightarrow T' = -k\lambda T \rightarrow T' + k\lambda T = 0$
- e. DE in space: $\frac{X''}{X} = -\lambda \rightarrow X'' = -\lambda X \rightarrow X'' + \lambda X = 0$

3. Eigenvalue analysis using boundary conditions.

- a. Assume $\lambda < 0$ thus: $X'' - \lambda X = 0 \rightarrow X = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \rightarrow$
 $X' = c_1 \sqrt{\lambda} e^{\sqrt{\lambda}x} - c_2 \sqrt{\lambda} e^{-\sqrt{\lambda}x}$
- BC: $X'(0) = c_1 \sqrt{\lambda} - c_2 \sqrt{\lambda} = 0 \rightarrow c_1 - c_2 = 0$
 - BC: $X'(L) = c_1 e^{\sqrt{\lambda}L} - c_2 e^{-\sqrt{\lambda}L} = 0$
 - This implies $c_1 = c_2 = 0 \rightarrow X(x) = 0 \rightarrow$ trivial
- b. Assume $\lambda = 0$ thus: $X'' = 0 \rightarrow X = c_1 x + c_2 \rightarrow X' = c_1 = a_0$
- BC: $X'(0) = X'(L) = c_1 = 0$
 - This implies $X(x) = c_2 \rightarrow$ not trivial!!!

c. Assume $\lambda < 0$ thus: $X'' + \lambda X = 0 \rightarrow X = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)$

- $X' = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}x) - c_1 \sqrt{\lambda} \sin(\sqrt{\lambda}x)$
- BC: $X'(0) = c_1 = 0 \rightarrow X = c_2 \cos(\sqrt{\lambda}x)$
- BC: $X'(L) = -c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}L) = 0$
- This implies $\sin(\sqrt{\lambda}L) = 0 \rightarrow \sqrt{\lambda}L = n\pi$ where $n = 1, 2, 3, \dots$
- This implies $\lambda = \left(\frac{n\pi}{L}\right)^2$ where $n = 1, 2, 3, \dots$
- By Superposition: $X(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$
 1. Note: I let $c_n \rightarrow a_n$
 2. Insulated ends gives us a cosine series

4. Now solve for $T(t)$.

a. Recall $T' + k\lambda T = 0 \rightarrow T(t) = b_1 e^{-k\lambda t} = b_1 e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

5. Now put it all together to get $u(x, t)$

a. $u(x, t) = X(x)T(t) \rightarrow a_0 + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$

- Note: for first term $\lambda = 0 \rightarrow e^{-k\lambda t} = 1$

6. Finally Apply Initial Condition

a. $u(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$

7. Example

a. PDE: $\frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2}$

b. BC: $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0$

c. IC: $u(x, 0) = (1-x)x^2$

d. General Solution applies with values $L = 1$ and $k = 5$ plugged in:

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right) = \boxed{a_0 + \sum_{n=1}^{\infty} a_n e^{-5(n\pi)^2 t} \cos(n\pi x)}$$

- What parts of the solution decay the fastest?
- What is the equilibrium solution?

e. Apply initial condition: $u(x, 0) = (1-x)x^2$

$$u(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n e^0 \cos(n\pi x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = (1-x)x^2$$

f. Now from Cosine series we have:

$$a_0 = \frac{1}{L} \int_0^L f(x) dx = \int_0^1 x^2(1-x) dx = \frac{1}{12}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(n\pi x) dx = 2 \int_0^1 x^2(1-x) \cos(n\pi x) dx$$

g. Plug this into TI:

$$a_n = -2 \frac{\cos(n\pi)}{(n\pi)^2} + 12 \frac{\cos(n\pi)}{(n\pi)^4} + 8 \frac{\sin(n\pi)}{(n\pi)^3} - \frac{12}{(n\pi)^4} \rightarrow$$

$$a_n = -\frac{2}{n^2 \pi^2} (-1)^n + \frac{12}{n^4 \pi^4} (-1)^n - \frac{12}{(n\pi)^4} \rightarrow$$

$$a_n = \frac{2}{n^2 \pi^2} (-1)^{n+1} + \frac{12}{n^4 \pi^4} ((-1)^n - 1)$$

h. Therefore:

$$u(x, t) = \frac{1}{12} + \sum_{n=1}^{\infty} a_n e^{-5(n\pi)^2 t} \cos(n\pi x)$$