SM286 Lecture Notes
Vibrating String with Fixed Ends

1. Wave Equation

a. PDE: $\frac{\partial^2 u}{\partial t^2} = \omega^2 \frac{\partial^2 u}{\partial x^2}$

b. BC: $u(0,t) = u(L,t) = 0$

c. IC: $u(x,0) = f(x)$ & $\frac{\partial u}{\partial t}(x,0) = g(x)$

2. Assume $u(x,t)$ is separable, i.e. $u(x,t) = \phi(x)T(t) \to u = \phi T$

a. Rewrite PDE as $u_{tt} = \omega^2 u_{xx} \to \phi T'' = \omega^2 \phi'' T$

b. Divide both sides by $\omega^2 \phi T \to \frac{\phi T''}{c^2 \phi T} = \omega^2 \frac{\phi'' T}{\phi T} \to \frac{1}{\omega^2}\frac{T''}{T} = \frac{\phi''}{\phi} = -\lambda$

- Note: Carry c^2 with T to simplify approach to final solution
- Note: λ must be a constant and is called an eigenvalue
- Note: Used $-\lambda$, again simplify approach to final solution

3. Eigenvalue analysis using boundary conditions.

a. DE in space: $\frac{\phi''}{\phi} = -\lambda \to \phi'' = -\lambda\phi \to \phi'' + \lambda\phi = 0$

b. Eigenvalue analysis is exactly as that for the heat equation with $T = 0$ ends

- $\lambda < 0$ → trivial solution
- $\lambda = 0$ → trivial solution
- $\lambda < 0$ → $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $\phi_n = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$

4. Now solve for $T(t)$.

a. DE in time: $\frac{1}{\omega^2}\frac{T''}{T} = -\lambda_n \to T'' = -\lambda_n \omega^2 T \to T'' + \lambda_n \omega^2 T = 0$

b. Let $\alpha_n^2 = \lambda_n \omega^2 = \left(\frac{n\pi\omega}{L}\right)^2 \to T'' + \alpha_n^2 T = 0$

c. Recall $\begin{cases} T'' + \alpha_n^2 T = 0 \to T_n = A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t) \\ T_n = A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t) \end{cases}$

5. Now put it all together to get $u(x,t)$

a. $u(x,t)=\phi(x)T(t)\rightarrow\sum_{n=1}^{\infty}c_n\sin\left(\sqrt{\lambda_n}x\right)\left(A_n\cos(\alpha_n t)+B_n\sin(\alpha_n t)\right)$

b. Combining constants yields: $u(x,t)=\sum_{n=1}^{\infty}\sin\left(\sqrt{\lambda_n}x\right)\left(A_n\cos(\alpha_n t)+B_n\sin(\alpha_n t)\right)$

6. Finally Apply Initial Condition

a.
$$\begin{cases} u(x,0)=\sum_{n=1}^{\infty}A_n\sin\left(\sqrt{\lambda_n}x\right)=f(x)\rightarrow \\ A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\sqrt{\lambda_n}x\right)dx \end{cases}$$

b.
$$\begin{cases} \frac{\partial u}{\partial t}(x,t)=\sum_{n=1}^{\infty}\alpha_n\sin\left(\sqrt{\lambda_n}x\right)\left(-A_n\sin.(\alpha_n t)+B_n\cos(\alpha_n t)\right)\rightarrow \\ \frac{\partial u}{\partial t}(x,0)=\sum_{n=1}^{\infty}\alpha_n B_n\sin\left(\sqrt{\lambda_n}x\right)\left(\cos(\alpha_n t)\right)=g(x)\rightarrow \\ \frac{\partial u}{\partial t}(x,0)=\sum_{n=1}^{\infty}\alpha_n B_n\sin\left(\sqrt{\lambda_n}x\right)=g(x)\rightarrow \\ \alpha_n B_n=\frac{2}{L}\int_0^L g(x)\sin\left(\sqrt{\lambda_n}x\right)dx\rightarrow B_n=\frac{2}{\alpha_n L}\int_0^L g(x)\sin\left(\sqrt{\lambda_n}x\right)dx \end{cases}$$

7. Substitute Back Values for λ_n and α_n

a.
$$\boxed{\begin{cases} u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left(A_n\cos\left(\frac{n\pi ct}{L}\right)+B_n\sin\left(\frac{n\pi ct}{L}\right)\right) \\ A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \& \quad \frac{2}{n\pi\omega}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}}$$

Example

b. PDE: $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$

c. BC: $u(0,t) = u(\pi,t) = 0$

d. IC: $u(x,0) = \begin{cases} x & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ \pi - x & \text{for} \quad \dfrac{\pi}{2} < x < \pi \end{cases}$ and $\dfrac{\partial u}{\partial t}(x,0) = g(x) = 0$

e. General Solution applies with values $L = \pi$ and $\omega^2 = 1$ plugged in:

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left(A_n \cos(nt) + B_n \sin(nt)\right) \\ A_n = \dfrac{2}{\pi}\left(\int_0^{\pi/2} x \sin(nx)dx + \int_{\pi/2}^{\pi} (\pi - x)\sin(nx)dx\right) \\ B_n = \dfrac{2}{n\pi\omega}\int_0^L g(x)\sin(nx)dx = 0 \end{cases}$$

f. Therefore: $\begin{cases} u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\cos(nt) \\ A_n = \dfrac{4}{n^2\pi}\sin\left(\dfrac{n\pi}{2}\right) \end{cases}$

8. Maple Demo – Modes of Vibration and Animation of Example.