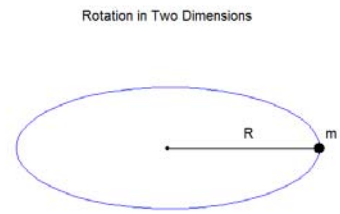

Particle in a Ring

<u>Derivation of the Wave Function</u>
Consider a particle of mass m that is rotating in a circular path with radius r. Polar coordinates are the logical choice to model this system. To solve this system on a quantum level, the Schrödinger equation must be expressed in polar coordinates:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathrm{r},\theta)+V(\mathrm{r},\theta)\Psi(\mathrm{r},\theta)=E\Psi(\mathrm{r},\theta). \quad (1)$$

Expanding the Laplacian and dropping the independent variables from Ψ:

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right)+V(\mathrm{r},\theta)\Psi=E\Psi. \quad (2)$$

The following stipulations are applied to the potential energy:

$$V(\mathrm{r},\theta)=\begin{cases}0 & r=R\\ \infty & r\neq R\end{cases}. \quad (3)$$

This implies that the radius is constant since the particle will not exist in a regime where the potential energy is infinite. Since radius is a constant $(r=R)$ and potential energy is zero, the Schrödinger equation is simplified:

$$-\frac{\hbar^2}{2mR^2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2}=E\Psi. \quad (4)$$

Note that partial derivative notation was removed since the system is now dependent on θ only.

The equation is rewritten as:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2}+\frac{2mR^2E}{\hbar^2}\Psi=0. \quad (5)$$

Using the definition of natural frequency $\omega=\sqrt{2mR^2E/\hbar^2}$ the equation is rewritten:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2}+\omega^2\Psi=0. \quad (6)$$

This is a second order homogeneous differential equation with constant coefficients whose solution is:

$$\Psi(\theta)=\mathrm{c}_1\mathrm{e}^{\mathrm{i}\omega\theta}+\mathrm{c}_2\mathrm{e}^{-\mathrm{i}\omega\theta}. \quad (7)$$

Since we are interested in the real solution only it is sufficient to let:

$$\Psi(\theta) = c_1 e^{i\omega\theta}. \tag{8}$$

Euler's formula states that:

$$\Psi(\theta) = c_1 e^{i\omega\theta} = c_1 \cos(\omega\theta) + i\sin(\omega\theta).$$

Let c_1 be a complex number (i.e. $c_1 = \alpha + \beta i$), then:

$$\Psi(\theta) = (\alpha + \beta i)e^{i\omega\theta} = (\alpha + \beta i)\cos(\omega\theta) + (\alpha + \beta i)i\sin(\omega\theta).$$

This equals:

$$\Psi(\theta) = \alpha\cos(\omega\theta) + \beta i\cos(\omega\theta) + \alpha i\sin(\omega\theta) - \beta\sin(\omega\theta).$$

Ignoring imaginary parts yields the familiar real form of the solution:

$$\Psi(\theta) = \alpha\cos(\omega\theta) - \beta\sin(\omega\theta).$$

To satisfy continuity, the wave function must satisfy a cyclic boundary condition $\Psi(\theta + 2\pi) = \Psi(\theta)$. Therefore:

$$e^{i\omega(\theta+2\pi)} = e^{i\omega\theta} \rightarrow e^{i\omega\theta}e^{2\pi i\omega} = e^{i\omega\theta} \rightarrow e^{2\pi i\omega} = 1 \tag{9}$$

This statement is true if and only if $\omega = \pm n$ where n is an integer. Therefore we rewrite the solution:

$$\Psi(\theta) = c_1 e^{in\theta}. \tag{10}$$

To solve for c_1, normalize the function:

$$\int_0^{2\pi} |\Psi(\theta)|^2 d\theta = \int_0^{2\pi} \Psi(\theta)\overline{\Psi(\theta)} d\theta = c_1^2 \int_0^{2\pi} \left(e^{in\theta}\right)\left(e^{-in\theta}\right) d\theta = 1, \tag{11}$$

$\overline{\Psi(\theta)}$ is the "***conjugate***" of $\Psi(\theta)$

Therefore:

$$c_1^2 \int_0^{2\pi} d\theta = 1 \longrightarrow c_1^2 2\pi = 1 \longrightarrow c_1 = \frac{1}{\sqrt{2\pi}} \tag{12}$$

yielding:

$$\boxed{\Psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}}. \tag{13}$$

Note that the value of n affects the speed and direction of rotation. Motion is counterclockwise for $n > 0$ and clockwise for $n < 0$. The phase angle φ is determined by the initial conditions.

Quantization of Energy

Recall:

$$\omega = \sqrt{\frac{2mR^2E}{\hbar^2}} \quad \text{and } \omega = n\,, \tag{14}$$

Therefore:

$$\frac{2mR^2E}{\hbar^2} = n^2 \longrightarrow \boxed{E_n = \frac{n^2\hbar^2}{2mR^2}}. \tag{15}$$

A table summarizing the first several energy states and corresponding wave functions appears below:

n	E_n	$\Psi_n(\theta)$
0	0	$\frac{1}{\sqrt{2\pi}}$
$\pm\mathbf{1}$	$\frac{\hbar^2}{2mR^2}$	$\frac{1}{\sqrt{2\pi}}e^{i\theta}$
$\pm\mathbf{2}$	$\frac{2\hbar^2}{mR^2}$	$\frac{1}{\sqrt{2\pi}}e^{2i\theta}$
$\pm\boldsymbol{n}$	$\frac{n^2\hbar^2}{2mR^2}$	$\frac{1}{\sqrt{2\pi}}e^{in\theta}$