The Gaussian Differential Equation → $y'' + (1 - x^2)y = 0$

<u>Express DE as a Power Series</u>

This is a homogeneous 2nd order differential equation complicated by the non-constant coefficients. We will solve this using power series technique. Assume the solution to the differential equation:

$$y = \sum_{n=0}^{\infty} c_n x^n \longrightarrow y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \longrightarrow y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}.$$

Therefore the differential equation can be rewritten as:

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+2} = 0.$$

<u>Shift Indices to Combine Summation Terms</u>

In the first summation let k=n-2 (which implies that n=k+2):

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \longrightarrow \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k.$$

In the second summation let n=k. In the third summation let k=n+2 (which implies that n=k-2):

$$\sum_{n=0}^{\infty} c_n x^{n+2} \longrightarrow \sum_{k=2}^{\infty} c_{k-2} x^k.$$

The differential equation is now rewritten in terms of k:

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k - \sum_{k=2}^{\infty} c_{k-2} x^k = 0.$$

We now strip of the first two terms of the first two summations:

$$2c_2 + 6c_3 x + \sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k + c_0 + c_1 x + \sum_{k=2}^{\infty} c_k x^k - \sum_{k=2}^{\infty} c_{k-2} x^k = 0.$$

The terms are combined as follows:

$$(c_0 + 2c_2) + (c_1 + 6c_3)x + \sum_{k=2}^{\infty} [c_{k+2}(k+2)(k+1) + c_k - c_{k-2}]\, x^k = 0.$$

From this we can conclude:

$$c_0 + 2c_2 = 0 \rightarrow \boldsymbol{c_2} = -\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{c_0},$$

$$c_1 + 6c_3 = 0 \rightarrow \boldsymbol{c_3} = -\frac{\mathbf{1}}{\mathbf{6}}\boldsymbol{c_1},$$

$$c_{k+2}(k+2)(k+1) + c_k - c_{k-2} = 0 \rightarrow \boldsymbol{c_{k+2}} = \frac{\boldsymbol{c_{k-2} - c_k}}{\boldsymbol{(k+1)(k+2)}} \text{ for k=2, 3, 4 …}$$

Now start with k=2 (since c_4 is the next term that we need) :

k=2: $c_4 = \frac{1}{12}(c_0 - c_2) = \frac{1}{12}\left(c_0 + \frac{1}{2}c_0\right) \longrightarrow \boxed{c_4 = \frac{1}{8}c_0,}$

k=3: $c_5 = \frac{1}{20}(c_1 - c_3) = \frac{1}{20}\left(c_1 + \frac{1}{6}c_1\right) \longrightarrow \boxed{c_5 = \frac{7}{120}c_1,}$

k=4: $c_6 = \frac{1}{30}(c_2 - c_4) = \frac{1}{30}\left(-\frac{1}{2}c_0 - \frac{1}{8}c_0\right) \longrightarrow \boxed{c_6 = -\frac{1}{48}c_0,}$

k=5: $c_7 = \frac{1}{42}(c_3 - c_5) = \frac{1}{42}\left(-\frac{1}{6}c_1 - \frac{7}{120}c_1\right) \longrightarrow \boxed{c_7 = -\frac{3}{560}c_1,}$

etc …

Therefore:

$$y \approx c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 \dots\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{3}{560}x^7 \dots\right)$$

Consider the special case where y(0)=1 and y'(0)=0. Note that:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} c_n x^n \longrightarrow y(0) = c_0 = 1,$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + \sum_{n=2}^{\infty} c_n n x^{n-1} \longrightarrow y'(0) = c_1 = 0.$$

Thus:

$$\boxed{y \approx 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 \dots}$$

Maclaurin Series Solution to e^x

If $f(x)$ the Maclaurin series can be written as $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$. Thus:

$$f(x) = e^x \to f(0) = 1,$$
$$f'(x) = e^x \to f'(0) = 1,$$
$$f''(x) = e^x \to f''(0) = 1,$$
$$etc \ldots$$

Thus:

$$e^x \approx \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots \to \boxed{e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \ldots}$$

Maclaurin Series Solution to $e^{-x^2/2}$

To obtain a Maclaurin series for $e^{-x^2/2}$, use the result for e^x replacing x with $-x^2/2$:

$$e^{-x^2/2} \approx 1 + \left(\frac{-x^2}{2}\right) + \frac{1}{2}\left(\frac{-x^2}{2}\right)^2 + \frac{1}{6}\left(\frac{-x^2}{2}\right)^3 \ldots \to \boxed{e^{-x^2/2} \approx 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 \ldots}$$

This seems to match the power series solution for the differential equation $y'' + (1 - x^2)y = 0$ when $y(0) = 1$ and $y'(0) = 0$.

Note that the original differential equation was second order suggesting that there is a second solution. Recall:

$$\boxed{y = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 \ldots\right) + \boldsymbol{c_1}\left(\boldsymbol{x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{3}{560}x^7 \ldots}\right)}$$

The **second polynomial** is the power series of the $\frac{1}{2}\sqrt{\pi}c_1 e^{-x^2/2}\mathrm{erfi}(x)$, where erfi(x) is the imaginary error function. We will not show this, but merely state it existence for completion.

Homework:

1. Show that $y = e^{-x^2/2}$ is a solution to the differential equation $y'' + (1 - x^2)y = 0$ and satisfies the initial values y(0)=1 and y'(0)=0.
2. What is the solution to the differential equation if y(0)=0 and y'(0)=1?
3. Add one additional term to each power series in the solution highlighted above