Example (Review)

- Solve:

$$\frac{d^2y}{dt^2} + y = 0,\ y(0) = 1,\ y'(0) = 1$$

- Solving DE;

$$(D^2 + 1)y = 0 \to D = \pm i \to y = c_1 \sin(x) + c_2 \cos(x)$$

- Applying 1st Initial Value:

$$y(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 = 1 \to\ y = c_1 \sin(x) + \cos(x)$$

- Applying 2nd Initial Value:

$$y' = c_1 \cos(x) - \sin(x) \to y'(0) = c_1 = 1 \to y = \sin(x) + \cos(x)$$

Example (Alternate Solution Using Complex Forms)

- Solve:

$$\frac{d^2y}{dt^2} + y = 0,\ y(0) = 1,\ y'(0) = 1$$

- Solving DE:

$$(D^2 + 1)y = 0 \to D = \pm i \to y = c_1 \mathrm{e}^{\mathrm{it}} + c_2 \mathrm{e}^{-\mathrm{it}}$$

This is consistent with how we handle real roots

- Applying 1st Initial Value:

$$y(0) = y = c_1 \mathrm{e}^0 + c_2 \mathrm{e}^0 = c_1 + c_2 = 1$$

- Applying 2nd Initial Value:

$$y' = c_1 \mathrm{i}\mathrm{e}^{\mathrm{it}} - c_2 i \mathrm{e}^{-\mathrm{it}} \to y'(0) = c_1 \mathrm{i} - c_2 i = 1 \to c_1 - c_2 = -i$$

- Two equations/two unknowns: $c_1 = \frac{1-i}{2}$ and $c_2 = \frac{1+i}{2}$.
- Final solution: $y = \frac{1-i}{2}\ \mathrm{e}^{\mathrm{it}} + \frac{1+i}{2} \mathrm{e}^{-\mathrm{it}}$

What is the Relationship between the Two Solutions?

- The above examples seem to imply that $\frac{1-i}{2}\ \mathrm{e}^{\mathrm{it}} + \frac{1+i}{2}\mathrm{e}^{-\mathrm{it}} = \sin(\mathrm{t}) + \cos\,(\mathrm{t})$
- The two are connected by Euler's Formula:

$$\boldsymbol{e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta)}$$

- i.e.

$$\frac{1-i}{2}\ \mathrm{e}^{\mathrm{it}} + \frac{1+i}{2}\mathrm{e}^{-\mathrm{it}}$$
$$= \left(\frac{1}{2}\cos(\mathrm{t}) + \frac{\mathrm{i}}{2}\sin(\mathrm{t})\right) + \left(-\frac{\mathrm{i}}{2}\cos(\mathrm{t}) + \frac{1}{2}\sin\,(\mathrm{t})\right) + \left(\frac{1}{2}\cos(\mathrm{t}) - \frac{\mathrm{i}}{2}\sin(\mathrm{t})\right)$$
$$+ \left(\frac{\mathrm{i}}{2}\cos(\mathrm{t}) + \frac{1}{2}\sin\,(\mathrm{t})\right) = \sin(\mathrm{t}) + \cos\,(\mathrm{t})$$

Example (the "Quantum Mechanics" way):

$$\frac{d^2y}{dt^2} + y = 0,\ y(0) = 1,\ y'(0) = 1$$

- Solving DE;

$y = c_1\mathrm{e}^{\mathrm{it}}$ (i.e. don't worry about the negative exponential)

- c_1 will be a complex variable. The real part of c_1 comes from the first initial value:

$$y(0) = c_1 = 1$$

- The imaginary part of c_1 comes from the second initial value:

$$y'(0) = c_1\mathrm{i} = 1 \rightarrow -c_1 = \mathrm{i} \rightarrow c_1 = -i$$

- Combine the two parts → $c_1 = 1 - i$
- Therefore: $y = (1-i)\mathrm{e}^{\mathrm{it}}$.
- Checking the solution in the original DE:
  - Note: $y' = (i+1)\mathrm{e}^{\mathrm{it}}$ and $y'' = (-1+i)\mathrm{e}^{\mathrm{it}}$
  - Therefore: $\mathrm{y}'' + \mathrm{y} = 0$

- Convert solutions using Euler's Equation:

$$y = (1 - i)\mathrm{e}^{\mathrm{it}} = (1 - \mathrm{i})(\cos(\mathrm{t}) + i\sin(\mathrm{t})) \rightarrow$$

$$y = \cos(t) + i\sin(t) - i\cos(t) + \sin(t) \rightarrow$$

$$y = \sin(\mathrm{t}) + \cos(t) + i\big(\sin(t) - \cos(t)\big)$$

- Now ignore the imaginary part: $y = \sin(\mathrm{t}) + \cos(t)$.

<u>Homework Problems (will be collected tomorrow!!)</u>:

1. Mike the Mathematician claims that the solution to a given differential equation is:

$$y = c_1\sin(t) + c_2\cos(t)$$

Pete the Physicists solves the same differential equation and claims the solution is:

$$y = Ae^{it} + Be^{-it}.$$

Using Euler's Formula express *A* and *B* as a function of c_1 and c_2 (i.e $A = f(c_1, c_2)$ and $B = g(c_1, c_2)$) demonstrating to Mike and Pete that they are indeed both correct.

2. Consider the DE:

$$\frac{d^2z}{dt^2} + \omega^2 z = 0,\ z(0) = 1,\ \ z'(0) = 0.$$

   a. Solve this DE using the auxiliary polynomial method (i.e. the method that you learned at the beginning of the course)
   b. Solve this DE using the "Quantum Mechanics Method".
   c. Show that the two solutions are equivalent.