The Hermite Differential Equation $\Rightarrow y'' - 2xy' + 2my = 0$

**Express DE as a Power Series**

This is a homogeneous 2\textsuperscript{nd} order differential equation with non-constant coefficients. Typically $m$ is a non-negative integer. We will solve this using power series technique. Assume the solution to the differential equation:

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}.$$

Therefore the differential equation can be rewritten as:

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - 2 \sum_{n=1}^{\infty} c_n n x^{n} + 2m \sum_{n=0}^{\infty} c_n x^{n} = 0.$$

**Shift Indices to Combine Summation Terms**

In the first summation let $k = n-2$ (which implies that $n=k+2$):

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} \quad \Rightarrow \quad \sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^{k}.$$

In the second and third summation let $n=k$. The differential equation is now rewritten in terms of $k$:

$$\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^{k} - 2 \sum_{k=1}^{\infty} c_k k x^{k} + 2m \sum_{k=0}^{\infty} c_k x^{k} = 0.$$

Note that we do not change the value of the second summation if $k$ starts at 0 vice 1, i.e.:

$$\sum_{k=0}^{\infty} c_k k x^{k} = \sum_{k=1}^{\infty} c_k k x^{k},$$

because the $k=0$ term is equal to 0. We now combine the terms as follows:

$$\sum_{k=2}^{\infty} [c_{k+2} (k+2)(k+1) + (2m - 2k)c_k] x^{k} = 0.$$
From this we conclude:

\[ c_{k+2}(k + 2)(k + 1) + (2m - 2k)c_k = 0, \]

Therefore:

\[ c_{k+2} = \frac{2(k - m)}{(k + 2)(k + 1)} c_k \text{ for } k = 0, 1, 2, 3 \ldots \]

**Apply Initial Conditions to Solve for Constants**

Given the initial conditions \( y(0)=a \), and \( y'(0)=b \), the values for \( c_0 \) and \( c_1 \) can be obtained as follows:

\[ y = \sum_{k=0}^{\infty} c_k x^k = c_0 + \sum_{k=1}^{\infty} c_k x^k \rightarrow y(0) = c_0 = a, \]

\[ y' = \sum_{k=1}^{\infty} c_k kx^{k-1} = c_1 + \sum_{k=2}^{\infty} c_k kx^{k-1} \rightarrow y'(0) = c_1 = b. \]

**Hermite Polynomials of Even Order**

Now consider the following initial conditions:

\[ y_m(0) = (-2)^m (m - 1)!! \rightarrow c_0 = (-2)^m (m - 1)!!, \]

\[ y'_m(0) = 0 \rightarrow c_1 = 0, \]

Here \( n!! \) is a double factorial term defined as follows:

\[ n!! = \begin{cases} (n)(n-2)(n-4) \ldots (1) & n \text{ odd,} \\ ((n)(n-2)(n-4) \ldots (2) & n \text{ even.} \]

For example: \( 7!!=(7)(5)(3)(1)=105. \)

From the recursion relationship above, we see that if \( c_1 = 0 \) then all \( c_k = 0 \) when \( k \) is odd. Now we find \( c_k \), for even values of \( k \).
For purposes of example, let \( m \) be an even integer, i.e. \( m=6 \). Then:

\[
c_0 = (-2)^3(5!!) = -8 \cdot 3 \cdot 5 = -120,
\]

\[
k = 0 \rightarrow c_2 = \frac{2(-6)}{(2)(1)} c_0 = -6c_0 = 720,
\]

\[
k = 2 \rightarrow c_4 = \frac{2(-4)}{(4)(3)} c_2 = -\frac{2}{3}c_2 = -480,
\]

\[
k = 4 \rightarrow c_6 = \frac{2(-2)}{(6)(5)} c_4 = -\frac{2}{15}c_4 = 64,
\]

\[
k = 6 \rightarrow c_6 = \frac{2(0)}{(8)(7)} c_4 = 0 \rightarrow c_8, c_{10}, c_{12} \ldots = 0.
\]

Therefore:

\[
y_6 = 64x^6 - 480x^4 + 720x^2 - 120.
\]

We test the solution by putting it back into the Hermite DE for \( m=6 \), i.e. \( y'' - 2xy' + 12y = 0 \). The first and second derivatives of \( y \) are:

\[
y_6' = 384x^5 - 1920x^3 + 1440x,
\]

\[
y_6'' = 1920x^4 - 5760x^2 + 1440.
\]

Substituting this into the DE yields:

\[
1920x^4 - 5760x^2 + 1440 - 2x(384x^5 - 1920x^3 + 1440x) + 12(64x^6 - 480x^4 + 720x^2 - 120) =
\]
\[
(-768+768) x^6 + (1920 + 3840 - 5760)x^4 + (-5760 - 2880 + 8640)x^2 + (1440 - 1440) = 0
\]

The solution above is called a Hermite polynomial of \textbf{order 6} and is denoted by \( H_6(x) \). Note that any multiple of this polynomial is also considered a Hermite polynomial of order 6. Hermite polynomials of other even valued orders can be obtained by using the same initial conditions and varying the values of \( m \) over the even numbers.
**Hermite Polynomials of Odd Order**

In order to obtain Hermite polynomials of odd order we specify the following initial conditions:

\[ y_m(0) = 0, \]
\[ y_m^{'}(0) = -(-2)^{(m+1)/2} (m!!), \]

which implies that \( c_0 = 0 \) and \( c_1 = -(-2)^{m+1}/2 \cdot m!! \). Then specify \( m \) to be any odd number, i.e. if \( m=5 \) then:

\[ c_1 = -(-2)^3(5!!) = 8 \cdot 3 \cdot 5 = 120, \]
\[ k = 1 \rightarrow c_3 = \frac{2(-4)}{(3)(2)} c_1 = -\frac{4}{3} c_1 = -160, \]
\[ k = 3 \rightarrow c_5 = \frac{2(-2)}{(5)(4)} c_3 = -\frac{1}{5} c_3 = 32, \]
\[ k = 5 \rightarrow c_7 = \frac{2(0)}{(7)(6)} c_5 = 0 \rightarrow c_9, c_{11}, c_{13} \ldots = 0. \]

Therefore:

\[ H_5(x) = 32x^5 - 160x^3 + 120x. \]
A summary of the first ten Hermite polynomials is listed below:

<table>
<thead>
<tr>
<th>Initial Conditions</th>
<th>$m$</th>
<th>$H_m(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Even Polynomials</strong></td>
<td>$y_m(0) = (-2)^m (m-1)!!$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4x^2 - 2$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$16x^4 - 48x^2 + 12$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$64x^6 - 480x^4 + 720x^2 - 120$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$</td>
<td></td>
</tr>
</tbody>
</table>

| **Odd Polynomials** | $y_m(0) = 0$ | $y'_m(0) = (-2)^{(m+1)/2}m!!$ |
| 1 | $2x$ |
| 3 | $8x^3 - 12x$ |
| 5 | $32x^5 - 160x^3 + 120x$ |
| 7 | $128x^7 - 1344x^5 + 3360x^3 - 1680x$ |
| 9 | $512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x$ |

**Hermite (physicists') Polynomials**
**Orthogonality Property of Hermite Polynomials**

A family of functions \( \{f_0(x), f_1(x), f_2(x), \ldots, f_n(x)\} \) is said to be orthogonal with respect to a weight \( w(x) \) over an interval \([a, b]\) if the following is true:

\[
\int_a^b f_m(x) f_n(x)w(x)dx = \begin{cases} 
0 & \text{for } m \neq n \\
\neq 0 & \text{for } m = n
\end{cases}
\]

Hermite polynomials form an orthogonal set of functions for the weight \( w(x) = e^{-x^2} \) over the interval \((−\infty, \infty)\). The exact relation is:

\[
\int_{-\infty}^{\infty} H_m(x) H_n(x)e^{-x^2} dx = \begin{cases} 
0 & m \neq n \\
\frac{2^n n! \sqrt{\pi}}{m} & m = n
\end{cases}
\]

This will not be proved, but can be demonstrated using any of the Hermite polynomials listed in the table. The property of orthogonality becomes important when solving the Harmonic oscillator problem.

**Homework**

1. Following the example for deriving \( H_6(x) \), derive \( H_4(x) \).

2. Verify \( H_4(x) \) by substituting it into the Hermite differential equation \( y'' - 2xy' + 2my = 0 \).

3. Calculate \( \int_{-\infty}^{\infty} H_4(x) H_4(x)e^{-x^2} dx \)
   
   a. Directly using your TI200.
   
   b. Indirectly using the Orthogonality Property above.

4. Calculate \( \int_{-\infty}^{\infty} H_3(x) H_4(x)e^{-x^2} dx \)
   
   a. Directly using your TI200.
   
   b. Indirectly using the Orthogonality Property above.