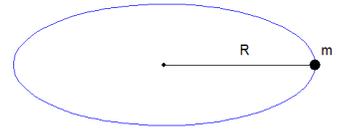


Particle in a Ring

Derivation of the Wave Function

Consider a particle of mass m that is rotating in a circular path with radius r . Polar coordinates are the logical choice to model this system. To solve this system on a quantum level, the Schrödinger equation must be expressed in polar coordinates:



$$-\frac{\hbar^2}{2m}\nabla^2\Psi(r, \theta) + V(r, \theta)\Psi(r, \theta) = E\Psi(r, \theta). \quad (1)$$

Expanding the Laplacian and dropping the independent variables from Ψ :

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right) + V(r, \theta)\Psi = E\Psi. \quad (2)$$

The following stipulations are applied to the potential energy:

$$V(r, \theta) = \begin{cases} 0 & r = R \\ \infty & r \neq R \end{cases}. \quad (3)$$

This implies that the radius is constant since the particle will not exist in a regime where the potential energy is infinite. Since radius is a constant ($r = R$) and potential energy is zero, the Schrödinger equation is simplified:

$$-\frac{\hbar^2}{2mR^2}\frac{d^2\Psi}{d\theta^2} = E\Psi. \quad (4)$$

Note that partial derivative notation was removed since the system is now dependent on θ only.

The equation is rewritten as:

$$\frac{d^2\Psi}{d\theta^2} + \frac{2mR^2E}{\hbar^2}\Psi = 0. \quad (5)$$

Using the definition of natural frequency $\omega = \sqrt{2mR^2E/\hbar^2}$ the equation is rewritten:

$$\frac{d^2\Psi}{d\theta^2} + \omega^2\Psi = 0. \quad (6)$$

This is a second order homogeneous differential equation with constant coefficients whose solution is:

$$\Psi(\theta) = c_1e^{i\omega\theta} + c_2e^{-i\omega\theta}. \quad (7)$$

Since we are interested in the real solution only it is sufficient to let:

$$\Psi(\theta) = c_1 e^{i\omega\theta}. \quad (8)$$

Euler's formula states that:

$$\Psi(\theta) = c_1 e^{i\omega\theta} = c_1 \cos(\omega\theta) + i \sin(\omega\theta).$$

Let c_1 be a complex number (i.e. $c_1 = \alpha + \beta i$), then:

$$\Psi(\theta) = (\alpha + \beta i) e^{i\omega\theta} = (\alpha + \beta i) \cos(\omega\theta) + (\alpha + \beta i) i \sin(\omega\theta).$$

This equals:

$$\Psi(\theta) = \alpha \cos(\omega\theta) + \beta i \cos(\omega\theta) + \alpha i \sin(\omega\theta) - \beta \sin(\omega\theta).$$

Ignoring imaginary parts yields the familiar real form of the solution:

$$\Psi(\theta) = \alpha \cos(\omega\theta) - \beta \sin(\omega\theta).$$

To satisfy continuity, the wave function must satisfy a cyclic boundary condition $\Psi(\theta + 2\pi) = \Psi(\theta)$. Therefore:

$$e^{i\omega(\theta+2\pi)} = e^{i\omega\theta} \rightarrow e^{i\omega\theta} e^{2\pi i\omega} = e^{i\omega\theta} \rightarrow e^{2\pi i\omega} = 1 \quad (9)$$

This statement is true if and only if $\omega = \pm n$ where n is an integer. Therefore we rewrite the solution:

$$\Psi(\theta) = c_1 e^{in\theta}. \quad (10)$$

To solve for c_1 , normalize the function:

$$\int_0^{2\pi} |\Psi(\theta)|^2 d\theta = \int_0^{2\pi} \Psi(\theta) \overline{\Psi(\theta)} d\theta = c_1^2 \int_0^{2\pi} (e^{in\theta})(e^{-in\theta}) d\theta = 1, \quad (11)$$

Therefore:

$\overline{\Psi(\theta)}$ is the "conjugate" of $\Psi(\theta)$

$$c_1^2 \int_0^{2\pi} d\theta = 1 \rightarrow c_1^2 2\pi = 1 \rightarrow c_1 = \frac{1}{\sqrt{2\pi}} \quad (12)$$

yielding:

$$\boxed{\Psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}}. \quad (13)$$

Note that the value of n affects the speed and direction of rotation. Motion is counterclockwise for $n > 0$ and clockwise for $n < 0$. The phase angle φ is determined by the initial conditions.

Quantization of Energy

Recall:

$$\omega = \sqrt{\frac{2mR^2E}{\hbar^2}} \quad \text{and} \quad \omega = n, \quad (14)$$

Therefore:

$$\frac{2mR^2E}{\hbar^2} = n^2 \quad \longrightarrow \quad \boxed{E_n = \frac{n^2\hbar^2}{2mR^2}}. \quad (15)$$

A table summarizing the first several energy states and corresponding wave functions appears below:

n	E_n	$\Psi_n(\theta)$
0	0	$\frac{1}{\sqrt{2\pi}}$
± 1	$\frac{\hbar^2}{2mR^2}$	$\frac{1}{\sqrt{2\pi}} e^{i\theta}$
± 2	$\frac{2\hbar^2}{mR^2}$	$\frac{1}{\sqrt{2\pi}} e^{2i\theta}$
$\pm n$	$\frac{n^2\hbar^2}{2mR^2}$	$\frac{1}{\sqrt{2\pi}} e^{in\theta}$